Theoretical Source Statistics of Thermo-Acoustic Instability within Reacting Flow

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Introduction and Objectives

Motivation

- Thermo-acoustic instabilities remain a problem within the combustion community.
- We quantify the source statistics of acoustic waves from flame kernels within premixed turbulent gases.
- Provide a first-principle statistical mathematical framework for assessing thermo-acoustic sources.

Objectives

- Formulate a closed-form integral equation for density, velocity, pressure, and temperature of wave fluctuations.
- Create a spectral prediction method for acoustic pressure from a reacting flow.

LES of Combustion Acoustic Source I

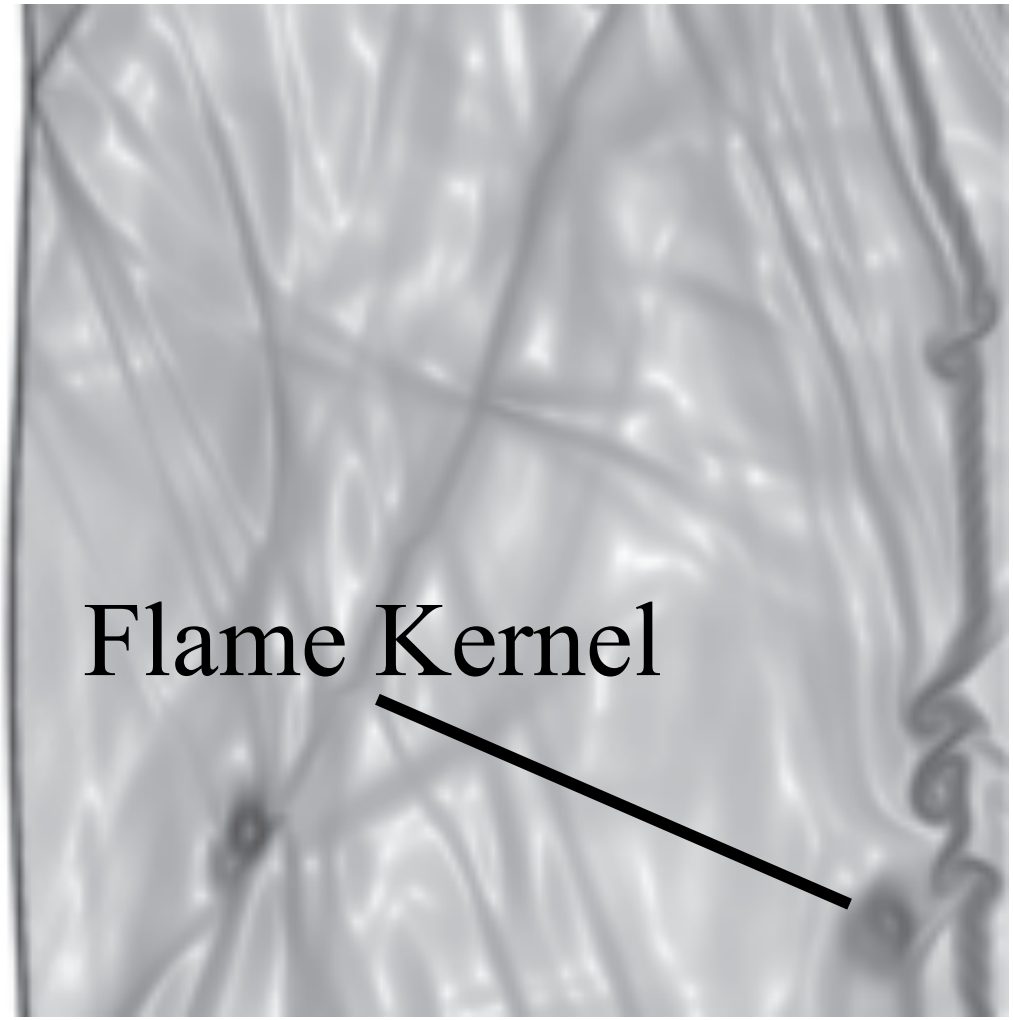

Figure 1: Time t_1. Contours of $|\nabla p|$ at four times illustrating the acoustic wave from a flame kernel within a premixed gas. Courtesy of Dr. Ryan Houim of the UF MAE Department.

LES of Combustion Acoustic Source II

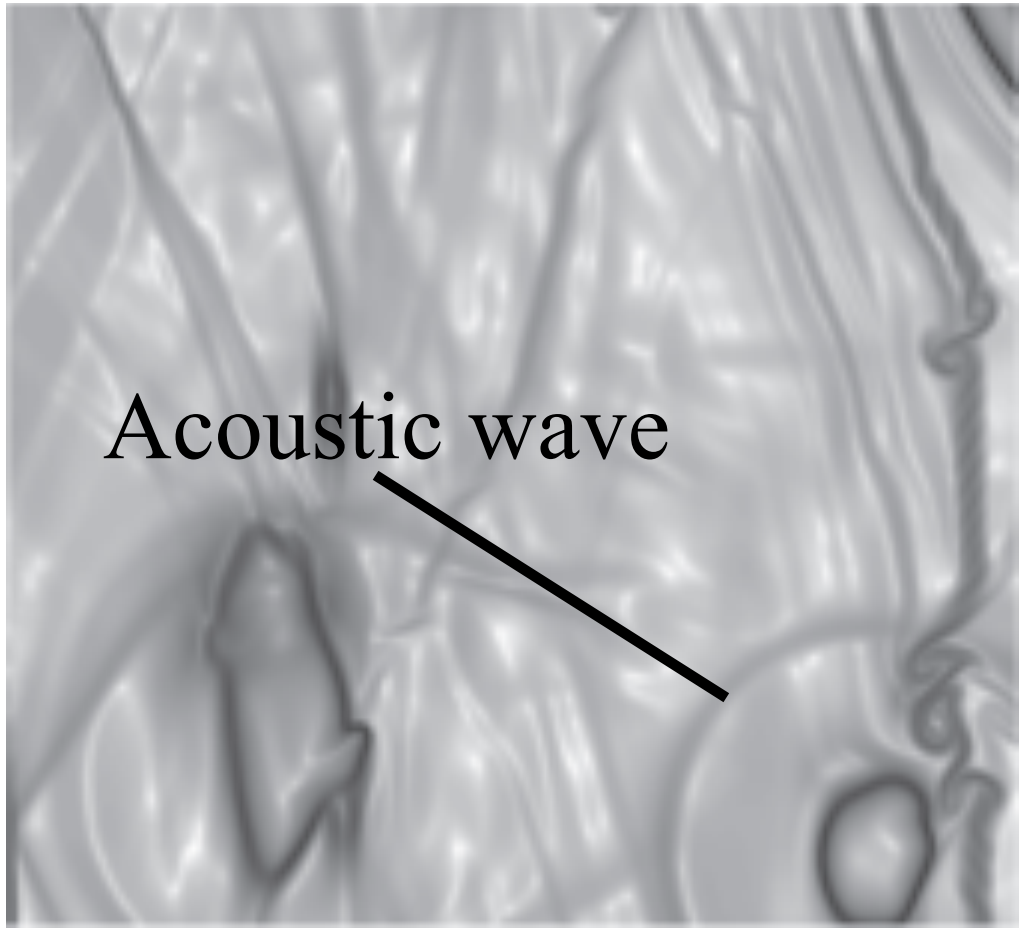

Figure 2: Time t_2. Contours of $|\nabla p|$ at four times illustrating the acoustic wave from a flame kernel within a premixed gas. Courtesy of Dr. Ryan Houim of the UF MAE Department.

LES of Combustion Acoustic Source III

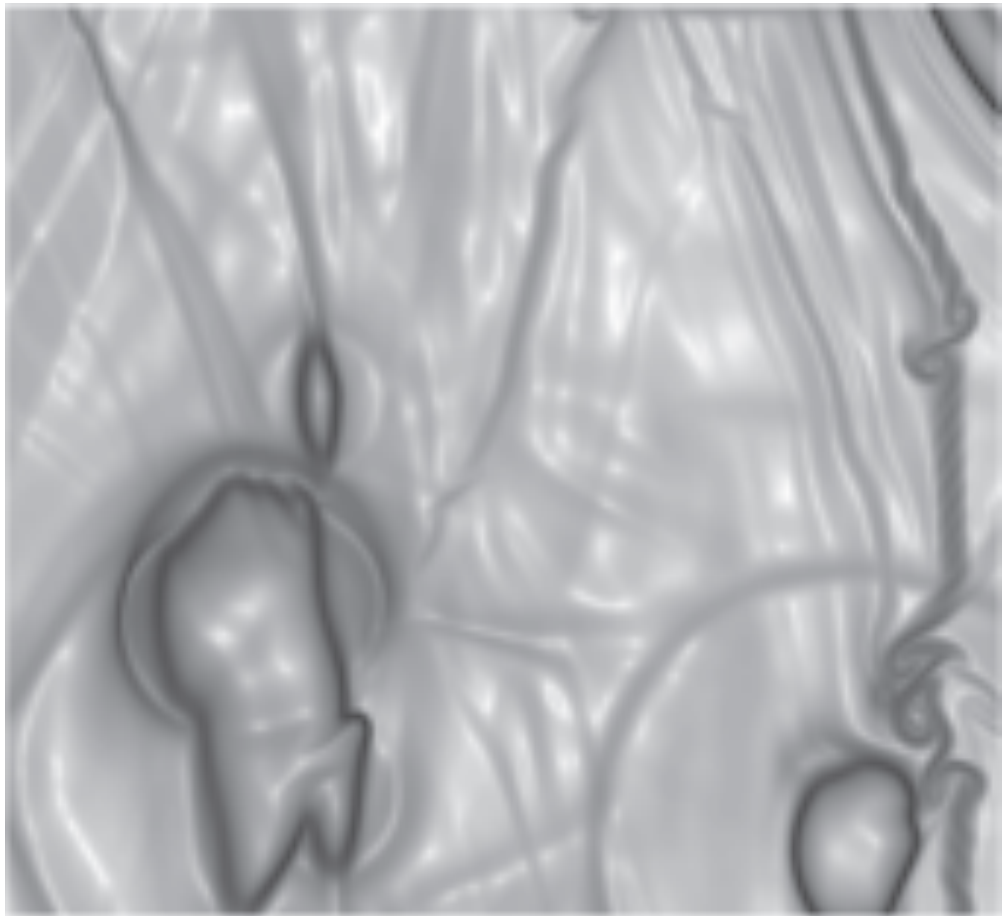

Figure 3: Time t_3. Contours of $|\nabla p|$ at four times illustrating the acoustic wave from a flame kernel within a premixed gas. Courtesy of Dr. Ryan Houim of the UF MAE Department.

LES of Combustion Acoustic Source IV

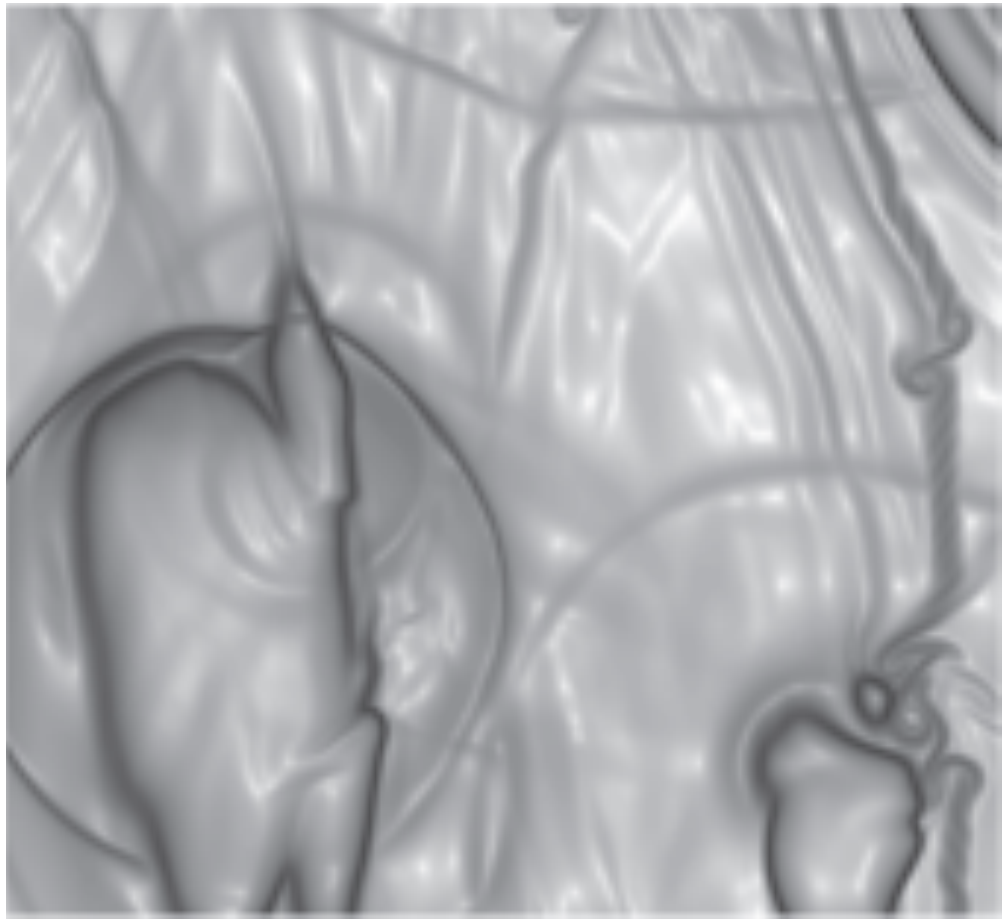

Figure 4: Time t_4. Contours of $|\nabla p|$ at four times illustrating the acoustic wave from a flame kernel within a premixed gas. Courtesy of Dr. Ryan Houim of the UF MAE Department.

Background – Traditional Combustion Noise Source Theory

Lieuwen et al. (1999) created an equation for noise from combustion, which is described as unsteady heat release.

$$p'(\boldsymbol{x}, t) = \frac{\gamma - 1}{4\pi c_\infty^2} \int \frac{1}{r} \frac{\partial q'}{\partial t} d\boldsymbol{y} \tag{1}$$

- Far-field assumption
- Assumes spherical spreading
- Additional sources of waves are not present

Background – Rayleigh

One condition proposed to predict the onset of combustion instability is the Rayleigh criterion. It involves the phased coupling of unsteady heat release with pressure.

"If heat be periodically communicated to, and abstracted from, a mass of air vibrating (for example) in a cylinder bounded by a piston, the effect produced will depend upon the phase of the vibration at which the transfer of heat takes place. If heat be given to the air at the moment of greatest condensation, or be taken from it at the moment of greatest rarefaction, the vibration is encouraged. On the other hand, if heat be given at the moment of greatest rarefaction, or abstracted at the moment of greatest condensation, the vibration is discouraged."

Lord Rayleigh (1878)

General Methodology and Procedure I

- A set of NSE, gas law, and mass fraction equations are order $N + 5$, where N is the number of species.
- Equations are
  - Continuity equation
  - Momentum equation with additional term $\rho \sum_{k=1}^{N} Y_k f_{k,i}$ for conservation of momentum, where Y_k is the mass fraction of species k and $f_{k,i}$ is the volume force acting on species k
  - Sensible energy equation e_s, with right hand side involving $\rho \sum_{k=1}^{N} h_{s,k} Y_k V_{k,i}$ and $\rho \sum_{k=1}^{N} Y_k f_{k,i} V_{k,i}$
  - Mass fraction equations

General Methodology and Procedure II

- Seek closed-form solution for the out-going fluctuating waves in the form of an integral equation.
- Let $\boldsymbol{q}$ be the vector of the field-variables consisting of $\rho, \boldsymbol{u}, p$, and Y_k, for $k = 1$ to N.
- Decompose equations with $\boldsymbol{q} = \overline{\boldsymbol{q}} + \tilde{\boldsymbol{q}} + \boldsymbol{q}'$
  - Overline represents the base flow or time-average
  - Tilde represents the fluctuating variable
  - Prime denotes the fluctuations about the base flow (overline) and turbulent combustion (tilde).
- Fluctuating quantities of outgoing waves are moved to the LHS of the equations and the remaining quantities are moved to the RHS.

General Methodology and Procedure III

- Solve equations for $\boldsymbol{q}'$ using method of the vector Green's function.
- Results in closed-form equation involving the vector Green's function of the linearized NSE including chemical reactions for each species k convolved with the NSE with chemical reactions acting on $\bar{\boldsymbol{q}}$ and $\tilde{\boldsymbol{q}}$.
- Form a spectral density equation for $\boldsymbol{q}$ by performing a two-point space-time cross-correlation of the source terms.
  - Statistical source formulation can be used to assess LES or DNS to detect the onset, magnitude, and exact source statistics of thermo-acoustic instabilities.

Source Terms of Waves

The vector components of the source $\mathbf{\Theta}_i$ are

- Θ_0 represents the source terms from the continuity equation.
- Θ_i (where $i = 1, 2, 3$) represents the three components of the source terms from momentum equations.
- Θ_4 represents the source terms from the energy equation.
- Θ_5 represents the source from the velocity-diffusion equation.

Sources Unique to Combustion Noise

We now define $\mathcal{T}^{(1)}_{\text{comb}}$ (components of Θ_i)

$$\mathcal{T}^{(1)}_{\text{comb}} = \Bigg[\underline{\rho}^{(1)} \sum_{n=1}^{N} \underline{Y}^{(1)}_k f^{(1)}_{k,i}, (\gamma-1)\underline{\dot{\omega}}^{\prime,(1)}_T, (\gamma-1)\underline{\rho}^{(1)} \sum_{k=1}^{N} \underline{Y}^{(1)}_k f^{(1)}_{k,i} \underline{V}^{(1)}_{k,i}, -\frac{\partial \underline{\rho}^{(1)} \underline{Y}^{(1)}_k}{\partial t}, \quad (2)$$

$$-\frac{\partial \underline{\rho}^{(1)} \underline{u}^{(1)}_j \underline{Y}^{(1)}_k}{\partial x_j}, \frac{\partial}{\partial x_j}\left(\underline{\rho}^{(1)} \underline{D}^{(1)}_k \frac{\underline{W}^{(1)}_k}{\underline{W}^{(1)}} \frac{\partial \underline{X}_k}{\partial x^{(1)}_j} \right), \underline{\dot{\omega}}^{(1)}_n \Bigg]^T$$

- Typically terms that will be dominant in aerospace flows will not involve viscous terms, body forces, or diffusion effects (D_k).
- $\mathcal{T}_{\text{Aero}}$ are the traditional aeroacoustic terms

Sources Unique to Combustion Noise

Customary to quantify statistical sources in the spectral domain.

We write the two-point space-time cross-correlations (of varying order depending on the number of terms) of the purely combustion source terms as

$$\left\langle \mathcal{T}^{(1)}_{\text{Comb}}, \mathcal{T}^{(2)}_{\text{Comb}} \right\rangle \tag{3}$$

Two-point cross-correlations of combustion terms with traditional aerodynamic terms are

$$\left\langle \mathcal{T}^{(1)}_{\text{Comb}}, \mathcal{T}^{(2)}_{\text{Aero}} \right\rangle \tag{4}$$

Rayleigh Criterion I

- Seek alternative forms of classic Rayleigh criterion, $\mathcal{R}$
- No terms involve p or $p\dot{q}$ within $\mathcal{T}^{(1)}_{\text{comb}}$
- Important terms appear in $\left\langle \mathcal{T}^{(1)}_{\text{Aero}}, \mathcal{T}^{(2)}_{\text{comb}} \right\rangle$

First term expanded, eliminate base flow of pressure with respect to time, and expand the second term

$$-\left\langle \sum_{i=1}^{3} \frac{\partial \hat{p}}{\partial x_j}\delta_{ij}, (\gamma-1)\hat{\dot{\omega}}'_T \right\rangle \text{ and } -\left\langle \frac{\partial \hat{p}}{\partial t}, (\gamma-1)\hat{\dot{\omega}}'_T \right\rangle \qquad (5)$$

- First term involves divergence of $\hat{p}$ with time derivative of the unsteady heat release, which is similar to the integrand of $\mathcal{R}$
- Second term involves time derivative of the fluctuation pressure $\hat{p}$ and time derivative of the unsteady heat release, which is also similar to $\mathcal{R}$

Rayleigh Criterion II

We propose the following Rayleigh criterion, as derived from the NSE and mass fraction equations

$$\boxed{\mathcal{R}_\alpha = \int_\Omega \left\langle \nabla \hat{p}, \hat{\dot{\omega}}'_T \right\rangle dV} \tag{6}$$

and

$$\boxed{\mathcal{R}_\beta = \int_\Omega \left\langle \frac{\partial \hat{p}}{\partial t}, \hat{\dot{\omega}}'_T \right\rangle dV} \tag{7}$$

These are integrals of two-point correlations

Traditional Acoustic Source I

The pressure due to unsteady heat release is found if terms within $\boldsymbol{\Theta}_i$ are set to zero except the unsteady heat release. Using a simplified vector Green's function

$$q'_k(\boldsymbol{x},t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=0}^{5} \mathcal{G}_{j,k}(\boldsymbol{x},t;\boldsymbol{y},\tau)\Theta_j(\boldsymbol{y},\tau)d\tau d\boldsymbol{y}. \tag{8}$$

Substituting in the vector Green's function and the source model, and seeking $k = 4$ for acoustic pressure we find

$$p'(\boldsymbol{x},t) \approx \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=0}^{5} \int_{-\infty}^{\infty} \left(-\frac{i\omega}{4\pi c_\infty^2 R} \exp\left[\frac{i\omega R}{c_\infty}\right]\right) \tag{9}$$
$$\times \exp\left[-i\omega\tau\right] \left((\gamma-1)\underline{\dot{\omega}}'_T\right) d\omega d\tau d\boldsymbol{y}.$$

Traditional Acoustic Source II

Only three unique combustion sources exist relative to the classical acoustic analogy.

If we eliminate the body force term, $f_{k,i}$ and the term involving $\underline{\dot{Q}}$ (external point heat source) then we find

$$\boxed{p'(\boldsymbol{x},t) \approx \int_{-\infty}^{\infty} \frac{\gamma-1}{4\pi c_\infty^2} |\boldsymbol{x}-\boldsymbol{y}|^{-1} \left(\overline{\dot{\omega}}'_T + \hat{\dot{\omega}}'_T\right) d\boldsymbol{y}} \tag{10}$$

If the base variable is eliminated then we have recovered the theory of Lieuwen et al. (1999), which is

$$p'(\boldsymbol{x},t) = \frac{\gamma-1}{4\pi c_\infty^2} \int \frac{1}{r} \frac{\partial q'}{\partial t} d\boldsymbol{y} \tag{11}$$

Summary and Conclusion

- Derived a closed-form integral equation for the thermo-acoustic field-variables from reacting flow
- Created a spectral prediction method for reacting flow
  - Two source terms are consistent with the Rayleigh instability model
  - Proposed Rayleigh instability model is unique and more general
- Created models involving two-point cross-correlations to quantify the strength of thermo-acoustic radiation from reacting flow
  - Two-point correlations account for both traditional aeroacoustic sources and combustion noise sources
  - Some source terms correspond to the unsteady heat release, which is uniquely a combustion-combustion source term

Acknowledgements

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Thank You

Questions?

Extra Slides

Abstract

We develop a statistical tool based on the equations of motion to quantify the various acoustic sources that cause thermo-acoustic instability. The Navier-Stokes equations, energy equation, and mass fraction equations are decomposed into their base, aerodynamic, and acoustic components. The acoustic components are solved for via the method of the vector Green?s function. The spectral densities of acoustic fluctuations are then formed. The resultant two-point cross-correlations of the Navier-Stokes operator, energy equation operator, and mass fraction operator on the base and aerodynamic flow-field yield thermo-acoustic statistical sources. Resultant sources are the two-point cross-correlations consisting of terms involving traditional aerodynamic interactions, aerodynamic-combustion interactions, and combustion-combustion interactions. We show that under certain simplifying assumptions, the traditional Rayleigh criterion and combustion acoustic equations are recoverable. We compare the newly proposed theory to traditional Rayleigh criterion and acoustic equations. The benefit of the theory is all thermo-acoustic sources are accounted for in a single two-point cross-correlation model, which is consistent with traditionally accepted theory.

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Sources Unique to Combustion Noise

We find the resultant combustion-acoustic source terms as

$$\boxed{(\gamma-1)^2\left\langle \underline{\dot{\omega}}_T^{\prime,(1)}, \underline{\dot{\omega}}_T^{\prime,(2)} \right\rangle}, -(\gamma-1)\left\langle \underline{\dot{\omega}}_T^{\prime,(1)}, \frac{\partial \underline{\rho}^{(2)} \underline{Y}_n^{(2)}}{\partial t} \right\rangle,$$

$$-(\gamma-1)\left\langle \underline{\dot{\omega}}_T^{\prime,(1)}, \frac{\partial \underline{\rho}^{(2)} \underline{u}_m^{(2)} \underline{Y}_n^{(2)}}{\partial x_m} \right\rangle,$$

$$\boxed{(\gamma-1)\left\langle \underline{\dot{\omega}}_T^{\prime,(1)}, \underline{\dot{\omega}}_n^{(2)} \right\rangle}, \left\langle \frac{\partial \underline{\rho}^{(1)} \underline{Y}_k^{(1)}}{\partial t}, \frac{\partial \underline{\rho}^{(2)} \underline{Y}_n^{(2)}}{\partial t} \right\rangle,$$

$$\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{Y}_k^{(1)}}{\partial t}, \frac{\partial \underline{\rho}^{(2)} \underline{u}_m^{(2)} \underline{Y}_n^{(2)}}{\partial x_m} \right\rangle, -\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{Y}_k^{(1)}}{\partial t}, \underline{\dot{\omega}}_n^{(2)} \right\rangle,$$

$$\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_j^{(1)} \underline{Y}_k^{(1)}}{\partial x_j}, \frac{\partial \underline{\rho}^{(2)} \underline{u}_m^{(2)} \underline{Y}_n^{(2)}}{\partial x_m} \right\rangle, -\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_j^{(1)} \underline{Y}_k^{(1)}}{\partial x_j}, \underline{\dot{\omega}}_n^{(2)} \right\rangle, \text{ and}$$

$$\boxed{\left\langle \underline{\dot{\omega}}_k^{(1)}, \underline{\dot{\omega}}_n^{(2)} \right\rangle}.$$

Two-Point Source Correlation Terms Involving Traditional Aeroacoustic Sources and Combustion Sources I

$$\boxed{-(\gamma-1)\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_i^{(1)} \underline{u}_j^{(1)}}{\partial x_j}, \underline{\dot{\omega}}_T^{\prime,(2)} \right\rangle}, \left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_i^{(1)} \underline{u}_j^{(1)}}{\partial x_j}, \frac{\partial \underline{\rho}^{(2)} \underline{Y}_n^{(2)}}{\partial t} \right\rangle,$$

$$\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_i^{(1)} \underline{u}_j^{(1)}}{\partial x_j}, \frac{\partial \underline{\rho}^{(2)} \underline{u}_m^{(2)} \underline{Y}_n^{(2)}}{\partial x_m} \right\rangle, \boxed{-\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_i^{(1)} \underline{u}_j^{(1)}}{\partial x_j}, \underline{\dot{\omega}}_n^{(2)} \right\rangle},$$

$$\boxed{-(\gamma-1)\left\langle \frac{\partial \underline{p}^{(1)}}{\partial x_j}\delta_{ij}, \underline{\dot{\omega}}_T^{\prime,(2)} \right\rangle}, \left\langle \frac{\partial \underline{p}^{(1)}}{\partial x_j}\delta_{ij}, \frac{\partial \underline{\rho}^{(2)} \underline{Y}_n^{(2)}}{\partial t} \right\rangle,$$

$$\left\langle \frac{\partial \underline{p}^{(1)}}{\partial x_j}\delta_{ij}, \frac{\partial \underline{\rho}^{(2)} \underline{u}_m^{(2)} \underline{Y}_n^{(2)}}{\partial x_m} \right\rangle, \boxed{-\left\langle \frac{\partial \underline{p}^{(1)}}{\partial x_j}\delta_{ij}, \underline{\dot{\omega}}_n^{(2)} \right\rangle},$$

$$\boxed{-(\gamma-1)\left\langle \frac{\partial \underline{p}^{(1)}}{\partial t}, \underline{\dot{\omega}}_T^{\prime,(2)} \right\rangle}, \left\langle \frac{\partial \underline{p}^{(1)}}{\partial t}, \frac{\partial \underline{\rho}^{(2)} \underline{Y}_n^{(2)}}{\partial t} \right\rangle,$$

Two-Point Source Correlation Terms Involving Traditional Aeroacoustic Sources and Combustion Sources II

$$\left\langle \frac{\partial \underline{p}^{(1)}}{\partial t}, \frac{\partial \underline{\rho}^{(2)} \underline{u}_m^{(2)} \underline{Y}_n^{(2)}}{\partial x_m} \right\rangle, \boxed{-\left\langle \frac{\partial \underline{p}^{(1)}}{\partial t}, \underline{\dot{\omega}}_n^{(2)} \right\rangle},$$

$$-\frac{(\gamma-1)^2}{2}\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_k^{(1)} \underline{u}_k^{(1)}}{\partial t}, \underline{\dot{\omega}}_T^{\prime,(2)} \right\rangle, \frac{\gamma-1}{2}\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_k^{(1)} \underline{u}_k^{(1)}}{\partial t}, \frac{\partial \underline{\rho}^{(2)} \underline{Y}_n^{(2)}}{\partial t} \right\rangle,$$

$$\frac{\gamma-1}{2}\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_k^{(1)} \underline{u}_k^{(1)}}{\partial t}, \frac{\partial \underline{\rho}^{(2)} \underline{u}_m^{(2)} \underline{Y}_n^{(2)}}{\partial x_m} \right\rangle, -\frac{\gamma-1}{2}\left\langle \frac{\partial \underline{\rho}^{(1)} \underline{u}_k^{(1)} \underline{u}_k^{(1)}}{\partial t}, \underline{\dot{\omega}}_n^{(2)} \right\rangle,$$

$$-(\gamma(\gamma-1))\left\langle \frac{\partial \underline{u}_j^{(1)} \underline{p}^{(1)}}{\partial x_j}, \underline{\dot{\omega}}_T^{\prime,(2)} \right\rangle, \gamma\left\langle \frac{\partial \underline{u}_j^{(1)} \underline{p}^{(1)}}{\partial x_j}, \frac{\partial \underline{\rho}^{(2)} \underline{Y}_n^{(2)}}{\partial t} \right\rangle,$$

$$\gamma\left\langle \frac{\partial \underline{u}_j^{(1)} \underline{p}^{(1)}}{\partial x_j}, \frac{\partial \underline{\rho}^{(2)} \underline{u}_m^{(2)} \underline{Y}_n^{(2)}}{\partial x_m} \right\rangle, -\gamma\left\langle \frac{\partial \underline{u}_j^{(1)} \underline{p}^{(1)}}{\partial x_j}, \underline{\dot{\omega}}_n^{(2)} \right\rangle,$$

Two-Point Source Correlation Terms Involving Traditional Aeroacoustic Sources and Combustion Sources III

$$-\frac{(\gamma-1)^2}{2}\left\langle \frac{\partial \underline{\rho}^{(1)}\underline{u}_j^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial x_j}, \underline{\dot{\omega}}_T^{\prime,(2)} \right\rangle, \frac{\gamma-1}{2}\left\langle \frac{\partial \underline{\rho}^{(1)}\underline{u}_j^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial x_j}, \frac{\partial \underline{\rho}^{(2)}\underline{Y}_n^{(2)}}{\partial t} \right\rangle,$$

$$\left\langle \frac{\gamma-1}{2}\frac{\partial \underline{\rho}^{(1)}\underline{u}_j^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial x_j}, \frac{\partial \underline{\rho}^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m} \right\rangle, -\left\langle \frac{\gamma-1}{2}\frac{\partial \underline{\rho}^{(1)}\underline{u}_j^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial x_j}, \underline{\dot{\omega}}_n^{(2)} \right\rangle,$$

$$(\gamma-1)^2\left\langle \underline{\dot{Q}}^{(1)}, \underline{\dot{\omega}}_T^{\prime,(2)} \right\rangle, -(\gamma-1)\left\langle \underline{\dot{Q}}^{(1)}, \frac{\partial \underline{\rho}^{(2)}\underline{Y}_n^{(2)}}{\partial t} \right\rangle,$$

$$-(\gamma-1)\left\langle \underline{\dot{Q}}^{(1)}, \frac{\partial \underline{\rho}^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m} \right\rangle \text{ and }, (\gamma-1)\left\langle \underline{\dot{Q}}^{(1)}, \underline{\dot{\omega}}_n^{(2)} \right\rangle$$

Background – Previous Instability Source Assessments

The Rayleigh criterion can be written as

$$\mathcal{R} = \int_{\Omega} \frac{p\dot{q}}{\overline{\rho} c_p \overline{T}} dV, \tag{12}$$

where $\dot{q}$ is the heat release per unit volume. $\mathcal{R}$ is a non-dimensional number, whose value when positive implies that there may be combustion instability.

There are many reacting flows where $\mathcal{R} >> 0$ and the flow remains stable.

Background - Previous Instability Source Assessments

Chu (1955) proposed a wave equation with heat source addition as

$$\frac{\partial^2}{\partial t}\left(\frac{p'}{\gamma p_\infty}\right) - c_\infty^2 \nabla^2 \left(\frac{p'}{\gamma p_\infty}\right) = \frac{\partial}{\partial t}\left(\frac{q}{c_p T_\infty}\right), \qquad (13)$$

where q is the rate of heat release per unit mass. Chu (1955) showed solutions of this equation using one-dimensional examples.

Background - Previous Instability Source Assessments

A solution in free-space is derived using the free-space Green's function of the wave equation

$$p'(\gamma p_\infty)^{-1} = (4\pi c_\infty^2)^{-1} \int r^{-1} \frac{\partial}{\partial t} \left[\frac{q(\xi, \eta, \zeta, t - rc_\infty^{-1})}{c_p T_\infty} \right] d\Omega, \quad (14)$$

where r is the distance from the source to observer. The solution for a point source is

$$p'(\gamma p_\infty)^{-1} = (4\pi c_\infty^2)^{-1} r^{-1} \frac{\partial}{\partial t} \left[\frac{Q(t - rc_\infty^{-1})}{c_p T_\infty} \right], \quad (15)$$

where $q = \delta Q$.

Background – Previous Instability Source Assessments

Chu (1956) published another important work on the stability of systems containing heat sources.

- Rayleigh criterion is reconsidered to assess stability of a flow-field with a heat source.
- A one-dimensional wave equation placed Q on the right hand side and related Q to the pressure.
- Relations of Q with p vary the stability of the system.
- Traditional instability analysis - exponential term is positive then the system is unstable.
- Careful choices of the relation between Q and p yield a system that results in a limit cycle.
  - Results in self-sustained vibrations of very large amplitude.

Traditional Aerodynamic Sources – Lighthill

- Lighthill (1952) proposed the acoustic analogy.
  - Rearrangement of the Navier-Stokes equations into a wave equation that is equated with equivalent sources.
- Noise from fully developed turbulence should scale as u^8.
- Compatible with traditional combustion theory, which involves a source term of unsteady heat release.
- Equivalent source technique implies source modeling.
- Only ascertained the aerodynamic fluctuations, which are highly dependent on $\rho u_i u_j$.

Traditional Aerodynamic Sources - FWH

- Ffowcs Williams and Hawkings (FWH) (1969) developed analogy to account for the noise from moving surfaces.
- Same approach as Lighthill but two additional concepts
  - Generalized derivatives.
  - Introduction of fictitious simple closed surface within the flow-field.
- Surface is defined by generalized functions.
- Practical applications and predictions – integration surface coincides with physical surface.
- Alternatively, surface is placed within a space that encompasses all sources.

Traditional Aerodynamic Sources – Farassat

- Farassat (2007) derived Formulation 1A to predict pressure in the far-field from the FWH surface.
- Represents most used and practical FWH solution and approach.
- We follow the formulation of Farassat (2007) and derive additional source terms that contain velocity fluctuations.
- Formulations are practical for the prediction of aerodynamic noise, but do not account explicitly for combustion.

Traditional Aerodynamic Sources – Farassat

One form of the solution via Farassat

$$
\begin{aligned}
4\pi p'(x,t) = \int_{-\infty}^{\infty} \Bigg[& \frac{\rho_\infty \dot{\mathcal{U}}_{n'}}{r(1-M_r)^2} + \frac{\rho_\infty \mathcal{U}_{n'}(r\dot{M}_r + c(M_r - M^2))}{r^2(1-M_r)^3} \\
& + \frac{\dot{\ell}_r}{cr(1-M_r)^2} + \frac{\ell_r - \ell_i \cdot M_i}{r^2(1-M_r)^2} + \frac{\ell_r(r\dot{M}_r + c(M_r - M^2))}{cr^2(1-M_r)^3} \\
& + \left(\frac{\dot{\rho}(\mathcal{V}_{n'} - \mathcal{U}_{n'}) + \rho(\dot{\mathcal{V}}_{n'} - \dot{\mathcal{U}}_{n'})}{r(1-M_r)^2} \right) (1 + \mathcal{M}_r) \\
& + \frac{\rho(\mathcal{V}_{n'} - \mathcal{U}_{n'})}{r(1-M_r)^2} \dot{\mathcal{M}}_r + \frac{\rho(\mathcal{V}_{n'} - \mathcal{U}_{n'})(\mathcal{V}_r - \mathcal{V}_i M_i)}{r^2(1-M_r)^2} \\
& + \frac{\rho(\mathcal{V}_{n'} - \mathcal{U}_{n'})(1 + \mathcal{M}_r)}{r^2(1-M_r)^3} (r\dot{M}_r + c(M_r - M^2)) \Bigg]_{ret} dS
\end{aligned}
\tag{16}
$$

Background – Source Studies I

- There is a large body of research involving the thermo-acoustic source and statistics of corresponding radiating waves.
- Many studies on the nature of turbulence with combustion that involve sound generation. Examples
  - Bray (1979)
  - Roberts and Leventhall (1973)
  - Clavin and Siggia (1991)
  - Kilham and Kirmani (1979)
  - Bailly et al. (2010)
  - Smith and Kilham (1963))

Background – Source Studies II

- Swaminathan et al. (2011), Liu (2015), and Liu et al. (2012) examined heat release, correlations, and sound radiation from turbulent premixed flames.
- Swirling flame studies of Wasle (2005) and Winkler et al. (2005) examined the heat release and acoustic efficiency of premixed flames.
- A number of Japanese studies by Kotake and Takamoto (1987), Kotake (1975), and Kotake and Takamoto (1990) examined combustion noise and its relation to the shape, the size of the burner nozzle, and chemical reactions.
- Pressure waves and their refraction was examined by Chu (1955)
- Pressure waves generated by a gaseous mixture by Strahle (1973).

On Radiated Combustion Noise

- Overviews of the statistics and radiation of acoustic waves from combustion - Dowling (1979), Strahle (1978), and Briffa et al. (1973).
- Klein et al. (1999) and Ihme et al. (2009) studied noise from non-premixed flames.
- Source studies of premixed flames from the aeroacoustics community (Jones (1979), Strahle and Shivashankara (1975), Hirsch et al. (2007), Hurle et al. (1968), and Rajaram and Lieuwen (2003,2009)).
- Notable models and prediction methods of noise from combustion (Wasle et al. (2009), Liu et al. (2014), and most recently Tam (2015)).
- A number of investigators examined instability in premixed combustion (Hurle et al. 1968; Price, Hurle & Sudgen 1968; Strahle 1978).

Mathematical Analysis

Mathematical Theory - Governing Equations I

The continuity equation is

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0,} \tag{17}$$

where u_j is the velocity, x is the spatial coordinate, t is the time, and ρ is the density. Alternatively, we can use the mass conservation for species k

$$\frac{\partial \rho Y_k}{\partial t} + \frac{\partial}{\partial x_j}\left[\rho\left(u_j + V_{k,j}\right) Y_k\right] = \dot{\omega}_k \tag{18}$$

where $V_{k,j}$ is the j component of the diffusion velocity V_k of species k, $\dot{\omega}_k$ is its associated reaction rate, and Y_k are the mass fractions of species k as $Y_k = m_k m^{-1}$. Here, m_k is the mass of species k and m is the total mass in the fluid parcel.

Mathematical Theory - Governing Equations I

The momentum equation is

$$\boxed{\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho \sum_{k=1}^{N} Y_k f_{k,i},} \tag{19}$$

where $f_{k,i}$ is the body force per unit volume on species k, k_{ij} is the Kronecker delta function, and p is the pressure. The shear stress, τ_{ij}, is

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}, \tag{20}$$

where μ is the viscosity.

Mathematical Theory - Governing Equations I

The energy equation for pressure is

$$\begin{aligned}
&\frac{\partial p}{\partial t} + \frac{\gamma - 1}{2}\frac{\partial \rho u_k u_k}{\partial t} + \frac{\gamma - 1}{2}\frac{\partial \rho u_j u_k u_k}{\partial x_j} \\
&\qquad = -\gamma \frac{\partial u_j p}{\partial x_j} + (\gamma - 1)\dot{\omega}'_T + (\gamma - 1)\frac{\partial}{\partial x_j}\left[\lambda \frac{\partial T}{\partial x_j}\right] \\
&+(\gamma - 1)\frac{\partial u_i \tau_{ij}}{\partial x_j} + (\gamma - 1)\dot{Q} + (\gamma - 1)\rho \sum_{k=1}^{N} Y_k f_{k,i} V_{k,i},
\end{aligned} \tag{21}$$

where $\dot{Q}$ is a heat source (e.g. plasma, spark, laser, etc.) and γ is the ratio of specific heats.

Mathematical Theory - Governing Equations I

The thermal conductivity, $\lambda = c_p \mu \mathrm{Pr}^{-1}$, where Pr is the Prandtl number. Also

$$\dot{\omega}'_T = -\sum_{k=1}^{N} h_k \dot{\omega}_k \tag{22}$$

where h_k is the the enthalpy of species k. We introduce the equation for h_k

$$h_k = h_{sk} + \Delta h^o_{f,k} = \int_{T_o}^{T} c_{p,k} dT + \Delta h^o_{f,k} \tag{23}$$

which is the summation of sensible enthalpy and chemical enthalpy. The diffusion velocity of species k is now

$$V_k = -D_k \frac{\nabla X_k}{X_k} + V_c = -D_k \frac{\nabla X_k}{X_k} + \frac{1}{W} \sum_{k=1}^{N} D_k W_k \nabla X_k \tag{24}$$

Mathematical Theory - Governing Equations II

The correction velocity is

$$V_c = \sum_{k=1}^{N} D_k \frac{W_k}{W} \frac{\partial X_k}{\partial x_i} \tag{25}$$

Diffusion velocities are governed by a system of equations shown by Williams (710 see NTC). We use the Hirschfelder and Curtiss approximation, which is an approximation of the exact system of Williams.

Mathematical Theory - Governing Equations I

The diffusion-velocity equation is

$$\boxed{\frac{\partial \rho Y_k}{\partial t} + \frac{\partial \rho u_j Y_k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\rho D_k \frac{W_k}{W}\frac{\partial X_k}{\partial x_j}\right) + \dot{\omega}_k} \tag{26}$$

where $\dot{\omega}_k$ is the reaction rate of species k.

Diffusion Coefficient

D_k is an equivalent diffusion coefficient of species k into the surrounding mixture

$$D_k = \frac{1 - Y_k}{\sum_{j \neq k} X_j / \mathcal{D}_{jk}} \tag{27}$$

and $X_k = Y_k W W_k^{-1}$ is the mole fraction of species k. W is the molecular weight of the mixture and W_k is the molecular weight of the species k, related by $W^{-1} = \sum_{k=1}^{N} Y_k W_k^{-1}$. $\mathcal{D}_{k,y}$ is the diffusion coefficient of species k into species y.

Vector Green's Function Notes

- Vector Green's function is written in terms of field-variable, $q_{k,g}^{\perp n} = \left[\rho_g^{\perp n}, \boldsymbol{u}_{i,g}^{\perp n}, p_g^{\perp n}, Y_{k,g}^{\perp n}\right]^T$.
- The new Green's function is $\mathcal{G}_k^n = \left[\rho_{\mathcal{G}}^n, u_{i,\mathcal{G}}^n, p_{\mathcal{G}}^n, Y_{k,\mathcal{G}}^m\right]^T$.
  - Subscript g denotes the Green's function
  - $\boldsymbol{x}$ and $\boldsymbol{y}$ represent the observer location and source location
  - τ is the source emission time.
- The refraction effects for a single stream jet in the sideline direction are minimal (see Tam 1998, Miller 2012, or Henry 2012).
- We assume that the environment is quiescent as the refraction effect of the shear layer is negligible.
- We perform the partial time derivative on the energy Green's Eqn. **??** and perform the gradient on the momentum Green's Eqn. **??**.
- We obtain an inhomogeneous wave equation.

Partial Solution of Vector Green's Function

After performing the Fourier transform, the solution to the vector Green's function for pressure can be found

$$\tilde{p}^n_{\mathcal{G}} = \left(\frac{i\omega}{4\pi c_\infty R} - \frac{1}{4\pi R^2} \right) \frac{(x_i - y_i)}{R} \exp\left[\frac{i\omega R}{c_\infty} \right] \delta_{in} - \frac{i\omega}{4\pi c_\infty^2 R} \exp\left[\frac{i\omega R}{c_\infty} \right] \delta_{4n}, \tag{28}$$

where $R = |\boldsymbol{x} - \boldsymbol{y}|$.

Rayleigh Criterion I

Notes on new terms

- Differ in their evaluation relative to traditional $\mathcal{R}$
- $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ involve the volumetric integration of a two-point cross-correlation.
- For each point within the reacting portion of the flow-field, every other point needs to be accounted for evaluation.
- Major advantage for the reacting flow – spatially coherent structures that involve highly coherent aerodynamic pressures and unsteady heat release due to reactants, *k*, will be powerful radiators of acoustic energy
- Highly negatively correlated *p* and unsteady heat release spatially may be stable (large negative values of $\mathcal{R}$)
- Evaluation of $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ result in a scalar value
- Like a critical Reynolds number for transition, there are critical values of $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ for combustion instability.

Rayleigh Criterion II

- Original approach of Rayleigh et al. are one-point theories, this approach uses whole spatial domain of reacting flow
- If a reacting flow reflects acoustic waves and they are of sufficient strength then the terms within Eqns. 6 and 7 must be modified to include the component of the reflected wave.
  - Occurs in unstable reacting flow undergoing thermo-acoustic instability.
  - For example, Eqn. 6 would have $\hat{p}$ approximated as $\hat{p} + p'$, but such an approach breaks a number of assumptions made for this method and will require a non-linear approach.
  - One can use Eqn. **??** to predict the acoustic spectra from these source terms, however, this will not yield any insight into the stability of a reacting flow, but only the radiated noise.

Instability Framework

- We assessed two-point correlations for combustion noise – but these are expensive to evaluate with LES
- Alternative – the two-point correlations can be modeled using approaches developed for aeroacoustics
  - Connect modeled sources via steady RANS
  - Leverage duct acoustic solvers for modal acoustics of NASA Langley – free to PI and have established connections
  - Acoustic solvers yield a Green's function required by our analysis
- This coupled approach (RANS, tailored GF, and instability theory), can yield a new kind of fast prediction approach for instability
  - This is similar to Poinsot's acoustic approach, which is based on the inhomogeneous Helmholtz equation with impedance boundary conditions

notes on talk - about 10 minutes total I

Contributed papers are limited to 10 minutes with 2 additional minutes for discussion. bring hdmi cable