Radiation from Large Scale Structures within a Turbulent Shear Layer

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Abstract
We examine the acoustic radiation from multiple high speed subsonic and supersonic free shear layers. We decompose the flow-field into a base component (an average), a component associated with the spatially and temporarily growing and decaying instability waves, and the acoustic radiation associated from the instability waves. We find an analytical solution for the acoustic radiation through the use of an acoustic analogy. The arguments of the acoustic analogy involve the two-point cross-correlation of quantities associated with the base flow and instability waves. The instability waves are modeled with a newly proposed basis function. A combination of LES, steady RANS solutions, and turbulence modeling are used to close the acoustic model. We compare our predictions to those of previous investigators and our predictions match previous theory. We find that the dominant acoustic radiation is due to the large-scale highly spatially coherent turbulence. The interaction of the instability waves cause secondary broadband radiation at higher observer angles.

Keywords
Turbulence, aeroacoustics, shear layer, acoustic analogy, noise

Introduction
High Reynolds number free shear flows are prevalent within aerospace applications and remain one canonical problem within the field of fluid dynamics. Shear layers are studied extensively to understand turbulence and associated acoustic radiation. Large-scale highly spatially coherent and temporally evolving turbulent structures convect at high speeds within high Reynolds number shear
layers. These shear layers radiate noise and similar turbulent structures exist in the more complicated jet flow. In this paper, we analyze and predict the acoustic radiation from high Reynolds number two-dimensional compressible shear layers using a decomposition approach and an acoustic analogy. Large-eddy simulation (LES) and steady Reynolds-averaged Navier-Stokes (RANS) simulations are used to decompose the flow-field and are the arguments of the acoustic analogy. We restrict our investigation to the sound radiation due to the instability waves, the instability waves interacting with one-another, and the instability waves interacting with the base flow.

**Experimental Research**

Mollo-Christensen performed many experimental investigations at MIT, which were designed to study and isolate large-scale coherent structures in turbulent flow-fields. In one such study, Mollo-Christensen (1967) investigated jet and shear flow instabilities with a focus on radiated jet and shear layer noise. Mollo-Christensen (1967) used two-point measurements and found correlations of velocity fluctuations as a function of jet and shear layer conditions isolated from the experimental apparatus. He was able to correlate the growth of instability waves with the correlations and radiated noise in the context of Lighthill’s acoustic analogy. In a similar study years later, Armstrong et al. (1977) performed pressure cross-spectra measurements within a jet plume and showed that large-scale structures exist in high Reynolds number flows. Armstrong et al. (1977) argued that low order large-scale structures dominant the flow and have relatively high acoustic radiation efficiency compared to other sound generation mechanisms.

A visualization technique was used by Crow and Champagne (1971) to examine orderly flow structures within round subsonic jets. It was shown that instabilities evolve from sinusoid to helical and finally a train of axisymmetric waves with increasing Reynolds number. By forcing the nozzle boundary layer, a wave forms that is predictable using traditional linear theory. A similar visualization technique was used by Dimotakis and Brown (1976), who examined a turbulent mixing layer in a water channel. Laser-Doppler velocimetry showed fundamental periodicity within the large-scale structures that contained correlation times longer than the integral time scale.

At high Mach number but low Reynolds number flow, the noise generation occurs from predominately instability waves, as shown by Morrison and McLaughlin (1979). Morrison and McLaughlin (1979) used a low-level excitation within the upstream portion of the flow-field that caused the majority of sound to be generated near the potential core, where the instability waves have the highest. Papamoschou and Roshko (1988) examined the growth rate and turbulent structures of a compressible plane shear layers in a range of Mach numbers, $\mathcal{M}$, from 0.2 to 4 using schlieren and Pitot probes. Papamoschou and Roshko (1988) showed that growth rates of the shear layer approach a value of approximately 0.2 for supersonic convective Mach numbers.

Suzuki and Colonius (2006a,b) examined instability waves within subsonic jets using a near-field phased array of microphones. Measurements were performed to ascertain instability waves through comparisons with eigenfunction reference solutions derived from linear theory. Suzuki and Colonius (2006a) showed that the instability waves evolved in the mean flow from the nozzle exit to the end of the potential core. Noise was shown to be highly correlated to the phased-array measurements and instability waves Suzuki and Colonius (2006b). In a recent and similar approach, Zaitsev et al. (2009) examined instability waves in supersonic jet flows using an azimuthal decomposition method developed.
Computational and Theoretical Research

We turn out attention to previous efforts of instability wave theory. For comprehensive reviews see Bradshaw (1977) and Huerre and Monkewitz (1990), who examined instabilities in mixing layers and jets. One of the first theoretical investigations was of Liu (1974), who studied large-scale wave like eddies or instabilities in turbulent shear layers. Liu (1974) decomposed the flow-field into components that consist of instability waves and fine-scale turbulence. He suggested that large-scale wavelike eddies are formed at the origin of the flow and amplification and decay are due to the mean turbulent shear. Mankbadi (1985) examined fundamental and subharmonic interactions by decomposing the flow into an average, highly coherent component, and additional random fluctuations. It was observed that as Strouhal number increased the peak location of the instability waves moves upstream and its integral time scale shortens.

Moore (1977) examined shear layer instabilities within jets and showed that there is a strong correlation which causes broadband noise even at subsonic Mach numbers. Moore (1977) excited the instability waves, which caused additional radiation, though this is the opposite relationship found by Morrison and McLaughlin (1979). Williams and Kempf (1978) also studied noise from large-scale structures within jets by modeling the turbulence as growing and decaying waves and paired vortices. They demonstrated a connection between the instability wave growth and decay with out-going Mach waves and their radiation angle. Crighton and Huerre (1990) examined Mach wave radiation through the use of the wave equation with boundary conditions dependent on instability waves. Suponitsky et al. (2010) performed numerical simulations of subsonic jets to examine how low wavenumber waves radiate sound. They showed that the low-frequency dominant noise in the downstream direction is due to instability waves and the fine-scale mixing noise is due to the breakdown of the flow structures into turbulence.

Goldstein and Leib (2005) examined non-parallel shear layers and showed that linear instability waves must be accounted for, unlike the shear layers with parallel mean flows that are present within Lilley’s acoustic analogy. By accounting for them, a casual solution of the problem can be obtained, which is unlikely possible within Lilley’s acoustic analogy. Of particular note is a study by Colonius et al. (1997), where they performed DNS and evaluated Lilley’s acoustic analogy. They argued that assumptions of the quadrupole nature of the source cannot entirely explain the radiated noise, and that the large-scale vortex pairing of the structures are better explained by a wavepacket approach. This study, amongst others, demonstrated discrepancies between predictions and measurements when the Reynolds number is high for subsonic flows.

Tam and Burton (1984a,b) examined sound generated by instability waves in two-dimensional shear layers and axisymmetric jets. They used mixed asymptotic expansions to construct a global solution, and can be used to predict the near-field and far-field sound. The work of Tam and Burton (1984a) is extended by Tam and Burton (1984b) to the axisymmetric jet case, where waves were decomposed into azimuthal modes. Their approach confirms that instability waves are an extremely important noise source within supersonic jet flows.

An emerging trend within the aeroacoustics community is to represent the large-scale growing and decaying instability waves with an empirical basis function known as the ‘wavepacket.’ A wavepacket is an empirical function that represents advecting disturbances that are correlated over distances on the order of the integral length scale of turbulence. The wavepacket function must be connected to the large-scale instability waves or other flow-structure within the flow. Often, the wavepacket model is connected to the outgoing acoustic radiation through the use of the convective wave equation. One example of this
approach is of Reba et al. (2010), who used a wavepacket to model noise from a jet. Reba et al. (2010) used an array of 80 microphones to calibrate the wavepacket acoustic radiation. They argued that large-scale wave-like structures are a dominant source of noise within these high speed jets at both subsonic and supersonic speeds. For complete reviews of wavepackets see Morris (2011) and Jordan and Colonius (2013).

Contributions of Professor Morris

Morris (1971) defended his Ph.D. dissertation under Geoffrey Lilley and its title was, ‘The Structure of Turbulent Shear Flow.’ Five years later, Morris (1976) examined the stability characteristics of an ideally expanded compressible axisymmetric jet. Numerical solutions showed that the critical Reynolds number increased as the jet meanflow profile becomes fully developed. A year later, Morris (1977) developed a model for large scale coherent structures of subsonic and supersonic axisymmetric jets. Fluctuations were divided into a mean, a periodic wave, and a small scale fluctuation, which is similar to the approach adopted in the present paper.

Tam and Morris (1980) examined the noise radiation of instability waves within compressible turbulent shear layers. They developed a solution that is valid in both the near- and far-field via asymptotic expansions of multiple scales. They predicted that the directivity pattern of acoustic radiation peaks near 20 degs. from the downstream axis, which agrees with the investigation of Papamoschou and Roshko (1988). Morris et al. (1990) returned to the problem of radiation from compressible shear layers a decade later, where he introduced a model for large-scale structures that depends on the superposition of instability waves. Finally, Morris (2009) proposed a method to predict acoustic pressure power spectral density in the far-field using the near-field fluctuations on a cylindrical surface surrounding the jet. The near-field was decomposed into large-scale similarity spectra and fine-scale similarity spectra according to the theory of Tam Tam et al. (1996, 2007) and the analysis of Viswanathan Viswanathan (2002). This decomposition was consistent with the the large-scale structures produced by instability waves or modeled via a wavepacket.

Present Approach

In this paper, we examine the acoustic radiation from multiple high speed subsonic and supersonic free shear layers. We decompose the flow-field into a base component (an average), a component associated with the spatially and temporarily growing and decaying instability waves, and the acoustic radiation associated from the instability waves. We find an analytical solution for the acoustic radiation through the use of an acoustic analogy developed by Miller (2017). The arguments of the acoustic analogy involve the two-point cross-correlation of quantities associated with the base flow, instability waves, and spectra of the field variables. The instability waves are modeled with a newly proposed basis function. A combination of large eddy simulations (LES), steady Reynolds-averaged Navier-Stokes (RANS) solutions, and models of turbulence are used to close the acoustic analogy. We compare our predictions to those of previous investigators.

The acoustic analogy and acoustic source modeling is partly based on the approach of Miller (2017). The decomposition of the flow-field is similar to that of Liu (1974) and Morris (1977). Our model of the instability waves is represented by an infinitely differentiable and integrable basis function that can be summed to represent more complex waves. Our approach differs greatly from those models relying
on wavepackets, where the wavepacket (see Jordan and Colonius (2013)) directly models the instability wave. Our model uses a basis function that is an argument of the two-point cross-correlation, which is used within the acoustic analogy.

In the next section, we present a mathematical model for the acoustic analogy and instability waves, which are representative of large coherent structures of growing and decaying turbulence. Next, results of the LES and steady RANS solutions of two-dimensional shear layers and their statistics are presented. We then present predictions for the spectral density of our model relative to the previous theories of Morris (Morris (1971, 1976, 1977); Tam and Morris (1980); Morris et al. (1990); Morris (2009)). Finally, we summarize our research and discuss its connection to more complicated flow-fields.

Mathematical Model

The Navier-Stokes equations govern the flow-field of the turbulent shear layer. The continuity equation is

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \tag{1}
\]

the momentum equation is

\[
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_j} \delta_{ij} + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{2}
\]

and the energy equation is

\[
\frac{\partial e_o}{\partial t} + \frac{\partial \rho u_j e_o}{\partial x_j} = -\frac{\partial u_j p}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \frac{\partial u_j \tau_{ij}}{\partial x_j}, \tag{3}
\]

where \(e_o\) is the total energy per unit mass, \(p\) is the pressure, \(u\) is the velocity, \(t\) is the time, \(x\) is the spatial coordinate, and \(\rho\) is the density. The shear stress is \(\tau_{ij} = 2\mu S^{*}_{ij}\), where \(S^{*}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij}\) and \(\delta_{ij}\) is the Kronecker delta function. Fourier’s law provides the heat flux \(q_j = -c_p \nu \mathcal{P} r^{-1} \frac{\partial T}{\partial x_j}\), where \(\mathcal{P} r\) is the Prandtl number, \(\mathcal{P} r = c_p \nu \lambda^{-1}\), and \(\lambda\) is the thermal conductivity. The total energy is related to the internal energy and kinetic energy by \(e_o = e + \frac{u_k u_k}{2}\). The flow is assumed to obey the ideal gas law and the relations \(\gamma = c_p c_v^{-1}\), \(p = \rho RT\), \(e = c_v T\), and \(R = c_p - c_v\) are used.

We decompose the field-variables (dependent variables) as

\[
q = \bar{q} + \tilde{q} + q', \tag{4}
\]

where \(q\) is the vector of the field-variables. The overbar operator represents the component of \(q\) that is the time-varying or time-averaged base flow. The tilde operator represents the component of turbulence that is highly spatially coherent and anisotropic. Finally, the prime operator denotes the radiating component due to fluctuations of highly spatially coherent and anisotropic turbulence. The decomposition of Eqn. 4 is substituted into the governing system of equations. The result is rearranged so that radiating quantities are on the left hand side, and the base flow and turbulent fluctuating quantities are on the right. We then eliminate terms on the left hand side of the decomposed equations that are high order, because radiating quantities are much smaller than non-radiating quantities. The right hand sides are rearranged.
and result in the exact Navier-Stokes equations and energy equation operators working on the base flow and fluctuating anisotropic turbulence. The resultant right hand side is considered as the source of noise and the left hand side is an operator that propagates acoustic radiation. The right hand side sources are then transformed using the newly proposed approach of Miller (2017) to involve scales of turbulence in time and space.

We now have a set of equations that can be solved through a convolution of the vector source term with the vector Green’s function of the linearized Navier-Stokes equations. The solution is written as

$$q'_k(x, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{n=0}^{3} q_{g,k,n}(x, t; y, \tau) \Theta_n(y, \tau) d\tau dy,$$

where the subscript $g$ denotes the Green’s function, $k = 0$ to 3 and represents which component of $q$ is sought, the subscript $n$ represents the vector component of the Green’s function, $x$ is the observer vector, $y$ is the source vector, $\Theta_n$ is the source term, and $\tau$ is the retarded time.

The vector Green’s function, $q_{g,k,n}(x, t; y, \tau)$, must satisfy the conservation of mass

$$\frac{\partial \rho_{g,n}}{\partial t} + \frac{\partial}{\partial x_j}[\rho_{g,n} \bar{u}_j + u_{i,g,n} \bar{p}] = \delta(x - y) \delta(t - \tau) \delta_{0n},$$

conservation of momentum

$$\frac{\partial}{\partial t}[\rho_{g,n} \bar{u}_i + u_{i,g,n} \bar{p}] + \frac{\partial}{\partial x_j}[\rho_{g,n} \bar{u}_i \bar{u}_j + u_{i,g,n} \bar{p} \bar{u}_j + u_{j,g,n} \bar{p} \bar{u}_i] - \bar{\mu} \frac{\partial}{\partial x_j} \left[ \frac{\partial u_{i,n}}{\partial x_i} + \frac{\partial u_{i,n}}{\partial x_i} \right]$$

$$+ \frac{2}{3} \bar{\pi} \frac{\partial^2 u_{k,n}}{\partial x_k} + \frac{\partial p_{g,n}}{\partial x_j} \delta_{ij} = \delta(x - y) \delta(t - \tau) \delta_{in},$$

and conservation of energy

$$\frac{\partial p_{g,n}}{\partial t} + \gamma - 1 \frac{\partial}{\partial t} \left[ \rho_{g,n} \bar{u}_k^2 + 2 u_{k,g,n} \bar{p} \bar{u}_k \right] + \gamma \frac{\partial}{\partial x_j} \left[ \rho_{g,n} \bar{u}_j + u_{j,g,n} \bar{p} \right]$$

$$+ \gamma - 1 \frac{\partial}{\partial x_j} \left[ \rho_{g,n} \bar{u}_k^2 + 2 u_{k,g,n} \bar{p} \bar{u}_k + 2 u_{k,g,n} \bar{p} \bar{u}_k \right] - \frac{\partial}{\partial x_j} \frac{cp\mu}{\rho R} \frac{\partial}{\partial x_j} \left[ \bar{p}_{g,n} \right]$$

$$- (\gamma - 1) \bar{\pi} \frac{\partial}{\partial x_j} \left[ \bar{u}_i \left( \frac{\partial u_{i,g,n}}{\partial x_j} + \frac{\partial u_{i,g,n}}{\partial x_i} \right) + u_{i,g,n} \left( \frac{\partial \pi_i}{\partial x_j} + \frac{\partial \pi_j}{\partial x_i} \right) \right]$$

$$+ \frac{2}{3} \delta_{ij} \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_{k,g,n}}{\partial x_k} + \mu \left( u_{i,g,n} \frac{\partial \pi_k}{\partial x_k} \right) \right) \right]$$

$$= \delta(x - y) \delta(t - \tau) \delta_{3n}.$$
conjugate, \( q^*_k, n \), to find an integral equation for the auto-correlation. We define the spectral density, \( S_k \), as the inverse Fourier transform of the auto-correlation

\[
S_k (x, \omega) = \int_{-\infty}^{\infty} \langle q_k (x, t) q_k (x, t + \tau^\dagger) \rangle \exp [i\omega \tau^\dagger] d\tau^\dagger,
\]

for \( k = 0 \) to 3 which corresponds to density, velocity components, and pressure, respectively. Using the general time-domain solution of Eq. 5 and the auto-correlation in the Wiener-Khinchin form. We now integrate in time, combine the resultant integrand within the inner summa, perform integration of \( \omega \) and \( \tau^\dagger \) (as they are the forward and inverse transforms and result in a prefactor), and finally simplify terms within the integral involving \( \tau \). An integral results that is the two-point space-time cross-correlation between \( \Theta_m \) and \( \Theta_n \). Let \( R_{m,n} (y, \eta, \tau) \) be the two-point space-time cross-correlation of the equivalent source

\[
R_{m,n} (y, \eta, \tau) = \langle \Theta_m (y, \tau) \Theta_n (y + \eta, \tau + \tau) \rangle = \int_{-\infty}^{\infty} \Theta_m (y, \tau) \Theta_n (y + \eta, \tau + \tau) d\tau.
\]

where \( \langle \rangle \) is the two-point space-time cross-correlation operator. Using this definition and Eqn. 10, we obtain the \( k \)th spectral density

\[
S_k (x, \omega) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{m=0}^{3} \sum_{n=0}^{3} q^*_k, m, n (x; y, \omega) q_k, n, m (x; y + \eta, \omega) R_{m,n} (y, \eta, \tau) d\tau d\eta dy.
\]

We now seek to model the source terms within \( R_{m,n} \) and write the source terms as a product of their magnitude and normalized two-point cross-correlation

\[
R_{m,n} \approx \{ R_{m,n} \} R,
\]

where \( R \) is the normalized two-point cross-correlation and \( \{ R_{m,n} \} \) is its magnitude, which will be modeled based on \( \Theta \).

The operator, \( \{ \} \), introduced in Eqn. 12, must approximate the magnitude of each two-point cross-correlation that varies spatially and temporarily in the flow-field. The magnitude is related to the products of the integrated spectra of the field-variables divided by appropriate length or time turbulent scales at corresponding wavenumbers. The magnitude of the fluctuating anisotropic component of turbulence is dependent on the wavenumber, \( \kappa \). For simplicity of modeling in this paper, we assume that each term of \( R_{m,n} \) is unique and that it has the same functional form. This approach was used successfully for isotropic turbulence, opposed to a highly spatially coherent anisotropic flow of this paper, by Miller (2017). The resultant source term is symmetric. For example, the magnitude of the source term involving \( < \Theta_1, \Theta_2 > \) is
\{ R_{(1),(2)} \} = \left( \frac{4 \mu (\ddot{u}_1 + \bar{u}_1)}{3l_{x,(1)}^2} + \frac{\mu (\ddot{v}_1 + \bar{v}_1)}{3l_{x,(1)}l_{y,(1)}} - \frac{\ddot{p}_1 + \bar{p}_1}{l_{x,(1)}} - \frac{(\ddot{p}_1 + \bar{p}_1)(\ddot{u}_1 + \bar{u}_1)}{l_{x,(1)}} \right) \\
+ \left( \frac{\mu (\ddot{u}_1 + \bar{u}_1)}{l_{y,(1)}^2} - \frac{(\ddot{p}_1 + \bar{p}_1)(\ddot{u}_1 + \bar{u}_1)(\ddot{v}_1 + \bar{v}_1)}{l_{y,(1)}} - \frac{(\ddot{p}_1 + \bar{p}_1)(\ddot{u}_1 + \bar{u}_1)}{\tau_{(1)}} \right) \\
\times \left( \frac{\mu (\ddot{v}_2 + \bar{v}_2)}{3l_{x,(2)}^2} + \frac{\mu (\ddot{u}_2 + \bar{u}_2)}{3l_{x,(2)}l_{y,(2)}} - \frac{(\ddot{p}_2 + \bar{p}_2)(\ddot{u}_2 + \bar{u}_2)(\ddot{v}_2 + \bar{v}_2)}{l_{x,(2)}} \right) \\
+ \frac{4 \mu (\ddot{v}_2 + \bar{v}_2)}{3l_{y,(2)}^2} - \frac{\ddot{p}_2 + \bar{p}_2}{l_{y,(2)}} - \frac{(\ddot{p}_2 + \bar{p}_2)(\ddot{u}_2 + \bar{u}_2)(\ddot{v}_2 + \bar{v}_2)}{l_{y,(2)}} - \frac{(\ddot{p}_2 + \bar{p}_2)(\ddot{u}_2 + \bar{u}_2)}{\tau_{(2)}} \right) .

(13)

Here, the ‘breve’ symbol represents the wavenumber spectra of the field-variable it operates on. The subscripts, (1) or (2), represent spatial positions within the flow-field separated by the vector \( \eta \). The appendix shows all ten terms within the source magnitude tensor. The wavenumber spectra are related to the breve symbols by

We relate the energy at \( \kappa \) to the energy spectrum by

\[
\ddot{u}_i (y, \tau) = \sqrt{\frac{2}{3}} \left( \int_{\kappa_1}^{\kappa_2} E_u(y, \kappa, \tau) d\kappa \right)^{1/2}.
\]

(14)

Correspondingly, we relate the spectrum of density at any point within the flow-field as

\[
\ddot{\rho} (y, \tau) = \left( \int_{\kappa_1}^{\kappa_2} E_\rho(y, \kappa, \tau) d\kappa \right)^{1/2}.
\]

(15)

Similarly, we relate the spectra of pressure, \( E_p \), on a wavenumber basis as

\[
\ddot{p} (y, \tau) = \left( \int_{\kappa_1}^{\kappa_2} E_p(y, \kappa, \tau) d\kappa \right)^{1/2}.
\]

(16)

Finally, the spectra of temperature, \( E_T \), is

\[
\ddot{T} (y, \tau) = \left( \int_{\kappa_1}^{\kappa_2} E_T(y, \kappa, \tau) d\kappa \right)^{1/2}.
\]

(17)

For any given wavenumber we choose the limits of integration to be constrained as \( \kappa_2 - \kappa_1 = c_\infty^{-1} \), to ensure that a power spectral density of acoustic pressure can be calculated. These spectra can either be modeled or calculated from an unsteady numerical simulation. We turn our attention to modeling the

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normalized two-point cross-correlation of the large-scale coherent anisotropic turbulence. We propose a model for \( R \)

\[
R = \int_{-\infty}^{\infty} \mathcal{P}_1(y_1, \Delta_1, \Delta_2, t) \mathcal{P}_2(y, \delta_1, \delta_2, \eta_1, \tau, t, \tau) dt.
\] (18)

Here, the growing and decaying instability waves are

\[
\mathcal{P}_\iota(y_\iota, \Delta_x, \Delta_y, t) = \exp \left[ -\frac{\Delta_\iota^2 y}{l_\iota^2} \right] \exp \left[ -\frac{t^2}{\tau_c^2} \right] \times \exp \left[ -\frac{(\Delta_x + y)^2}{l_x^2} \right] \exp \left[ i(\kappa(\Delta_x + y) - \kappa t u_c) \right],
\] (19)

where \( u_c \) is the local convection velocity and \( \Delta \) represents a vector component in the \( x \) or \( y \) direction, which the instability wave model is dependent on. Here, we replace \( (\iota) \) with \( (1) \) or \( (2) \) depending on which source position \( \mathcal{P} \) is at. We isolate terms within the model for \( \mathcal{P} \) involving time and integrate with respect to time in Eqn. 18. We find the expression

\[
\pi^{1/2} \exp \left[ -4\pi i \left( \kappa(1)^2 - \kappa(2)^2 \right) - \frac{1}{4} \kappa(1)^2 \right] \exp \left[ \frac{1}{4} \kappa(2)^2 \right] \times \mathcal{R}_{m,n}(y, \eta_1, \tau_1) \right] dt.
\] (20)

Isolating terms involving \( \tau \) and integrating yields

\[
\pi \tau_c(1) \tau_c(2) \exp \left[ -\frac{1}{4} \kappa(1)^2 \right] \exp \left[ -\frac{1}{4} \kappa(2)^2 \right] \times \mathcal{R}_{m,n}(y, \eta_1, \tau_1) \right] dt.
\] (21)

Note that there are two values of \( \kappa \) at positions separated by \( \eta \). We now simplify the model equation for the spectral density of acoustic pressure \( (k = 3) \) radiated from a two-dimensional shear layer consisting of highly coherent anisotropic turbulence. The spectral density is

\[
S_k(x, \omega) = \int_{-\infty}^{\infty} \sum_{m=0}^{3} \sum_{n=0}^{3} q_{g,k,m}(x; y, \omega) \mathcal{Q}_{g,k,n}(x; y + \eta, \omega) \times \pi \tau(1) \tau(2) \exp \left[ -\frac{1}{4} \kappa(1)^2 \right] \exp \left[ -\frac{1}{4} \kappa(2)^2 \right] \times \mathcal{R}_{m,n}(y, \eta_1, \tau_1) \right] dt dy.
\] (22)

Arguments have been proposed for the source term within model Eq. 11. The vector Green’s function \( q_{g,k,m} \) must be found. For this purpose, we assume that the environment is quiescent except for a region that contains the two-dimensional turbulent shear layer. As the sound field is dominated by waves from large-scale structures, the refraction effects within the shear layer are small. Miller (2017)
shows the analytical solution for the linearized Navier-Stokes vector Green’s function within a quiescent environment. This assumption implies that the refraction effects upon sound propagation are neglected. We use this vector Green’s function here without modification.

**Computational Fluid Dynamics**

For the purposes of evaluating the newly developed acoustic analogy and for studying the turbulent structure, we performed a series of simulations consisting of both RANS and LES. The following sections discusses those approaches.

**Steady Reynolds-Averaged Navier-Stokes**

Two-dimensional steady RANS solutions are obtained at shear layer Mach numbers of 0.75, 1.25, and 1.75 with the commercial software package FLUENT. Figure 1 shows the computational domain and boundary conditions used for the steady RANS solver. The computational domain is consistent with the problem description of Tam and Morris (1980). The domain extends 4.5 m in the streamwise direction and 2 m in the cross-stream direction. The flows are separated by a virtual plate on the negative $x$-axis that extends 0.5 m upstream from the origin. This results in two flows entering the domain from the left boundary into the domain. A velocity inlet boundary condition is specified on the upper left inlet and a static pressure inlet is specified on the lower left side to simulate a quiescent fluid. Symmetric boundary conditions are used on the upper boundary and along the negative $x$-axis. The remaining boundaries are modeled as static pressure outlets to avoid shock wave formation.

An implicit density based solver is used in conjunction with Roe flux-vector splitting to solve the steady RANS equations. The RANS equations are closed by an $k$-$\epsilon$ model implementation proposed by Launder and Spalding (1974). A second order upwind scheme is utilized for terms involving quantities of turbulence. Convergence acceleration is achieved by dividing the mesh into two-parts, where the upper half is approximated with the inlet velocity before the simulation commences.

**Large-Eddy Simulation using HiFiLES**

We adapt and modify an LES code developed by Castonguay et al. (2011) called High Fidelity Large Eddy Simulation (HiFiLES). HiFiLES is a high order of accuracy compressible viscous flow solver that runs on unstructured computational domains. The code uses the discontinuous Galerkin (DG) (Hesthaven and Warburton (2007)) method with an energy-stable flux reconstruction scheme known as Vincent-Castonguay-Jameson-Huynh (VCJH) (Vincent et al. (2011)) to recover the solution. A non-linearly stable Runge-Kutta fourth order of accuracy and five step method is used to advance the algorithm in time. The code is highly parallel lending to the element-local nature of the scheme. We perform our LES simulations in parallel after using Parallel Graph Partitioning and Fill-reducing Matrix Ordering (ParMETIS) (Karypis and Kumar (1999)) for mesh partitioning.

The accurate prediction of many scales of turbulence and broadband acoustic radiation is critical for our LES simulation of the two-dimensional shear layer. We require high-order discretization methods within our LES code. HiFiLES is a high-order method that yields a significant advantage over low-order accuracy approaches which do not necessarily capture as large a range of scales. Furthermore, high-order methods are inherently less dissipative, resulting in less interference with the correct development of the
turbulent energy cascade. Finally, our selection of a high order accuracy method results in significantly less computational expense opposed to an equivalent second order method.

LES depends heavily on flux reconstruction. The domain is partitioned into \( n \) non-overlapping elements. Each element in the physical domain is mapped onto a standard element. This involves a transformation from the coordinate system used in the computational domain to the standard element, which uses the Jacobian. The HiFiLES code uses \( p^{th} \) order polynomials to approximate the solution and flux. The solution and flux extrapolated to the flux points from the solution points are not continuous on the element interfaces. We use a Riemann solver for the DG method and which are then corrected using the VCJH scheme.

LES relies on the philosophy of numerically resolving the large-scale turbulent structures and modeling or implicitly dissipating the small-scale structures. The equations of motion are filtered for this purpose. For example, the filtered momentum equation is

\[
\frac{\partial \tilde{\rho} \tilde{u}}{\partial t} + \frac{\partial}{\partial x_j} (\tilde{\rho} \tilde{u}_i \tilde{u}_j) = \frac{\partial}{\partial x_j} (-\tilde{p} \delta_{ij} + \tilde{\tau}_{ij} - \tau_{ij}^{SGS})
\]

(23)

and the filtered shear stress tensor is

\[
\tilde{\tau}_{ij} = 2 \mu \left( \dot{S}_{ij} - \frac{1}{3} \dot{S}_{kk} \delta_{ij} \right)
\]

(24)

where \( \dot{S}_{ij} \) is the filtered rate-of-strain tensor and \( \tau_{ij}^{SGS} \) is the sub-grid scale stress. A sub-grid scale model must be used for \( \tau_{ij}^{SGS} \). We adopt the sub-grid scale model that is based on the eddy viscosity assumption. The sub-grid term is modeled as \( \tau_{ij}^{SGS} = 2 \mu_t \dot{S}_{ji} \). In the present work, we use a basic Smagorinsky model as the sub-grid scale model. The Smagorinsky models the eddy viscosity, \( \mu_t \), as \( \mu_t = C_s^2 \rho \Delta^2 |\dot{S}| \), where \(|\dot{S}| = (2 \dot{S}_{ij} \dot{S}_{ij})^{1/2} \).

**LES of the Two-Dimensional Shear Layer** The computational grid used in the LES simulation of the \( M = 0.75 \) two-dimensional free shear-layer is shown in Fig. 2. The computational domain contains 67,600 grid points, consisting of 520 in the streamwise direction and 130 in the cross-stream direction. Alone the negative \( x \)-axis is a horizontal separation of the domain that is used to generate the shear layer and separate the two incoming flows. The grid point spacing near the origin of the computational domain, which is the origin of the shear layer, has a grid point spacing of \( 1 \times 10^{-4} \) m. The inlet of the domain is -3 m upstream from the origin of the shear layer to allow the flow-field to develop. The outlet of the domain is extended 100 m downstream from the origin and the 50 m in the cross-stream direction. In the downstream direction, the domain is long and wide enough to prevent the boundary from influencing the flow upstream.

Figure 3 shows the boundary conditions prescribed for the LES simulation. At the inlet, which is the left side of the computational domain, the velocity is prescribed along with density, temperature, and Mach number. We allow for \( M_\infty > 0 \) and at the upper inlet and at the lower inlet \( (y < 0) \) we set \( M_\infty \approx 0 \). Recall that the inlet extends 5 m. in the cross-stream direction and is long enough to eliminate any wall-based flow-effects that might adversely alter the shear layer development. A slip-wall boundary condition is prescribed at the upper and lower boundaries of the computational domain. Recall that the two flows are separated along the negative \( x \)-axis, and this is enforced by prescribing a slip wall boundary condition at these grid points. This strategy prevents the development of a boundary layer that would alter
the desired dynamics of the shear layer. The boundary on the right side of the domain, where the flow exits the computational domain, are non-reflective and characteristic, to simulate the ambient environment. There are reflections of acoustic signals from the solid boundaries. However, since our method of noise prediction takes fluctuating values of aerodynamics arguments as acoustic sources, the weak reflection of acoustic signals from the solid boundaries will only have negligible effect on the final noise prediction result, unlike Ffowcs-Williams Hawkings (FWH) solvers (Williams and Hawkings (1969)).

At the beginning of each LES simulation, the flow-field is initialized by running with polynomial solution representation of order 0 within each element for 2 millions time steps. Then, the polynomial is altered to order one and another 8 million time steps are performed until a steady state is achieved. The simulation is run for another 1 million time steps at polynomial order three. At this point, the LES is run for additional steps to sample the turbulent of statistics. Simulation parameters for each portion of the LES are shown in Table 1. The non-dimensional time is $t^{\triangle} = t U_{\text{ref}} L_{\text{ref}}^{-1}$, where $t$ is the physical time, $L_{\text{ref}}$ is the reference length, and $U_{\text{ref}}$ is the reference velocity. For this LES, we let $L_{\text{ref}} = 1$ m and $U_{\text{ref}} = 347.128$ m s$^{-1}$.

We developed a new sampling subroutine to obtain time-varying statistics and wavenumber spectra. The field-variables at user selected points are extracted and stored in text files. This is a three-step process. The first step consists of specifying which points within the flow-field will be probed. Second, the locations of the probe points in the reference domain are obtained from a perspective transformation. During the final step, the desired variables at the probe points are written to the separate files as the simulation runs.

Results

We now presents results from our steady RANS solutions, LES, and acoustic analysis. LES results are shown for $M = 0.75$ and steady RANS results are shown for $M = 0.75$, 1.25, and 1.75. Our acoustic analysis shows the basis functions for large-scale anisotropic turbulence and select acoustic radiation predictions.

Steady RANS Solutions

Figure 4 shows Steady RANS values of $k$ and $\epsilon$ along the $x$-axis for $M = 0.75$, 1.25, and 1.75. Note there are two $y$-axes scales. Both $k$ and $\epsilon$ increase with increasing Mach number. Large values of $k$ and $\epsilon$ occur just downstream of the origin of the shear layer. At the shear layer origin both turbulence quantities exhibit large spikes in their magnitude and eventually start decaying with increasing streamwise position.

The length scales and time scales of turbulence must be estimated based on the RANS solution when an LES solution is not present. This is performed through the relations, $l_x = l_c k^{3/2}/\epsilon$ and $\tau = \tau_c k/\epsilon$, where $l_x$ is the streamwise length scale, $l_c$ is a coefficient, $\tau_c$ is a coefficient, and $\tau$ is the time scale. The cross-stream length scale is estimated as one-third of the streamwise length scale. These estimates for the three Mach numbers examined are shown in Fig. 5. Near the origin of the shear layer they all take on the value of zero. In the initial development of the shear layer they rise at a non-linear rate and eventually increase linearly with increasing streamwise position. This is to be expected and matches many measurements and the LES results presented in the next section.
**Large-Eddy Simulation**

Numerical probes in the computational domain are used to collect time history of field variables at the locations of interest every 100 time steps. This enabled us to resolve the fluctuation of field variables while taking up much less space compared to the snapshots of the whole flow field. The data collected is used to calculate the power spectra, turbulent scales, and convection velocity. Figure 6 shows contours of vorticity and pressure near the origin of the shear layer. Similar structures can be seen in the contour plot of density as shown in Fig. 7. The spatially evolving turbulent vortices and radiating acoustic waves are clearly present within the LES simulation.

A qualitative comparison of the LES meanflow with the empirical formula proposed by Tam and Morris (1980). For our \( M = 0.75 \) shear layer LES, we find that the spreading rate is 0.18, which is larger than that found by Tam and Morris (1980). Recall that the LES simulation is two-dimensional and the vortices predicted are more coherent and have longer life-times than equivalent three-dimensional vortices. A mean streamwise velocity profile is shown in Fig. 8. The \( x \)-axis is the non-dimensional coordinate \( \eta = (y - y_{0.5})(\epsilon(x - x_0))^{-1} \), where \( y_{0.5} \) is the position where the mean streamwise velocity component is half the free-stream value. The variable \( x_0 = 0 \) and \( \epsilon \) are the spreading rate defined as \( \epsilon = (y_{0.9} - y_{0.1})(x - x_0)^{-1} \). The LES captures the trend of the variation of the mean velocity in the cross-stream direction and shows a good self-preservation along the stream-wise direction. The difference between the LES result and the empirical relation of Tam and Morris (1980) is due to the over estimation of the shear layer spreading rate.

Wavenumber spectra are calculated throughout the LES computational domain. We show one spectrum in Fig. 9 for illustrative purposes. The spectrum is located at \( x = 0.0446 \) m, and the Kolmogorov -5/3 law is shown as a dashed line. The \( x \)-axis represents wavenumber, \( \kappa = f u_c^{-1} \), where \( f \) is frequency and \( u_c \) is streamwise convection velocity (see Yule (1972)).

We estimate the integral time scale at each sampling point as

\[
T = \int_0^\infty \frac{\langle u(x,t)u(x,t+\tau)\rangle}{\langle u^2 \rangle} d\tau,
\]

where \( T \) is the integral time scale of the streamwise velocity component. Auto-correlation functions within the LES simulation show large periodic oscillation. Integrations of Eqn. 25 are restriction in the range from \( \tau = 0 \) to the first zero crossing for each sampling point. Spatial scales are derived from temporal scales using Taylor’s frozen turbulence hypothesis (see Tennekes and Lumley (1972) and Cenedese et al. (1991) for details). The length scale is derived from the time scale as \( l_x = u_c T \), where \( u_c \) is the wavelength independent overall convection velocity of the flow. It is obtained from the two-point cross-correlation (McColgan and Larson (1977)).

**Aeroacoustic Results**

A central point of this research is the use of a basis function of the instability wave shown in Eqn. 19 for \( \mathcal{P} \). We show the behavior of this equation for typical values within a turbulent flow. Figure 10 shows the normalized instability wave basis function at various times, \( t \), on the \( x \)-axis. To illustrate their behavior, we select arguments of Eqn. 19 as \( \kappa = 10.0, u_c = 0.50, l_x = 1.0, l_y = 1.0, \) and \( \tau = 1.0 \). The arguments are held constant while \( t \) varies from the time which the instability wave is at its largest amplitude \( (t = 0) \) through its decay at \( t = 1 \). The \( x \)-axis is in meters and the \( y \)-axis is the normalized amplitude. The
envelope represents the maximum bound of the instability wave in space-time. It is Gaussian in shape because we chose $\iota_c = 1$ for simplicity. Figure 10 shows that as time increases the instability wave decreases, but the wavenumber and propagation speed are altered. In the context of this model, instability waves reside everywhere within the turbulent flow and their amplitudes, wavenumbers, and propagation speeds vary.

Contours of a single instability wave modeled by Eqn. 19 are shown in Fig. 11. Here, the $x$-axis and $y$-axis are in meters and contours of the instability wave model are from negative one to one. The arguments of the model are $\kappa = 10$, $u_c = 0.50$, $l_x = 2$, $l_y = 0.5$, and $t = 0$. Figure 10 showed the temporal and spatial variation of the instability wave model along the central axis, here, we emphasize the spatial variation. The length scales within Eqn. 19 control its spatial width and lengths, that are dependent on the wavenumber and the integral scale of turbulence. The periodic wave structures travel at the convection velocity $u_c$ that is a vector. It is in the $x$-direction for illustration purposes only. There is little disturbance from the instability wave model far from its origin. At large or small times the disturbance due to the instability wave is negligible. The flow is considered statistically stationary in the framework of the model while the instability waves are transient.

Acoustic radiation occurs through two-mechanisms if the source consists of instability waves. Within the context of the present acoustic analogy, the first type of radiation is due to each instability wave. This is represented through an instability wave source separation distance of zero, which results in a single instability wave such as that shown in Fig. 11. The second type is when two instability waves interact with each other. This situation is illustrated in Fig. 12. Here, the contours are normalized magnitude of $P_1$ interacting with $P_2$. Each instability wave possess its own unique set of properties and this is reflected in the present prediction approach. In Fig. 12, the arguments of instability wave (1) and (2) are $\kappa(1) = 10$, $\kappa(2) = 20$, $u_{c,(1)} = 0.50$, $u_{c,(2)} = 0.6$, $l_x = 1$, $l_y = 0.25$, $\tau = 1$, and $t = 0$. The two instability waves are at different positions located at $x(1) = 1$ and $y(1) = 0$ and $x(1) = 2$ and $y(1) = 0.25$, respectively. The interaction of two instability waves of different wavenumbers and spatial positions results in a very unique flow-field and acoustic radiation. Combinations of $\{R_{m,n}^k\}$ for each instability wave results in an aeroacoustic model for the superposition of all possible instability waves within the shear layer. By definition of the model, the combination of the all instability waves results in the total spectral ‘energy’ of each field-variable.

We now turn our attention towards acoustic predictions. These predictions now use either the steady RANS or the LES results, unlike Figs. 10 through 12 that used simple prescribed values to illustrate the wave basis function behavior. Figure 13 shows far-field SPL directivity as a function of radiation angle, $\theta$, from the downstream direction. Predictions of Tam and Morris (1980) are shown at $M = 0.75$, 1.25, and 1.75 as black, red, and blue lines with symbols, respectively. Corresponding predictions of the present method are shown at the same Mach numbers and are based upon the steady RANS solution estimations of length scales, time scales, convection velocities, and field-variable spectra. The LES based prediction is shown as a single line without a symbol, and it contains much more variance relative to the RANS based method or predictions of Tam and Morris (1980). These predictions correspond to the low frequency of $\omega = 0.005$ and are normalized by the maximum predicted SPL. Generally, the predictions are in good agreement given the large range of the $y$-axis and the estimations involved in the RANS based predictions.
The peak magnitude of the prediction occurs near 20 deg., and this is expected for two-dimensional shear layer radiation. The radiated noise is broadband in nature and contains two main peaks. The large-scale coherent anisotropic structures are responsible for radiating noise predominantly in the downstream direction and also contribute to a higher radiation angle near 65 deg. We see that both the predictions of Tam and Morris (1980) and ours capture this trend. Tam and Morris (1980) write, “only those wavenumber components of the pressure fluctuation at the edge of the flow field which have a sonic phase velocity to some location in the far field can radiate noise.” This is an excellent explanation of why as \( \mathcal{M} \) increases the dominant broadband radiated noise rises in the peak radiation direction of 20 deg. The primary broad lobe of noise, centered near 20 deg., is due to the large-scale anisotropic turbulent structures within the shear layer. The secondary broad lobe of noise, centered at the higher angle near 65 deg., is due to the interaction of two large-scale anisotropic turbulent structures located at two different spatial locations. This latter situation is illustrated in Fig. 12. In the context of a single model that only contains large-scale turbulent structures, we have predicted both the primary and secondary broad lobe seamlessly.

We examine the variation of the far-field SPL directivity as a function of frequency, \( \omega \), and radiation angle, \( \theta \). Figure 14 shows predictions based on the steady RANS data at \( \mathcal{M} = 1.75 \) and frequencies \( \omega = 0.005, 0.015, \) and 0.050. Predictions of Tam and Morris (1980) are also shown. The directivity is normalized by the maximum value of the prediction at \( \omega = 0.005 \). Tam and Morris (1980) wrote, “As can be seen from ... the directivity pattern in the far field is governed by the wavenumber component spectrum amplitude as a function of wavenumber and a \( \sin^2 \theta \) weighting factor.” Predictions show a lower decay of SPL at the peak radiation angle of 20 deg. and more broad spectral content.

A predicted spectrum of the present approach is shown in Fig. 15. The \( y \)-axis is SPL per unit Hz and is normalized by the maximum predicted SPL. The \( x \)-axis is normalized frequency, where the maximum frequency of the prediction is approximately 3 kHz. The observer angle is \( \theta = 20 \) deg. The large-scale similarity (LSS) spectrum and small-scale similarity (SSS) spectrum of Tam and Seiner Tam et al. (1996) are normalized by corresponding peak SPL and corresponding maximum frequency, \( f_{\text{max}} \). The SSS represents an empirical acoustic spectrum from three-dimensional jet turbulence that fits a wide range of measurement data, especially at the sideline direction of the jet. It represents the noise contribution due to relatively incoherent locally isotropic turbulence. The LSS spectrum is similar to the SSS spectrum except it represents the portion of the spectrum that is due to highly spatially coherent anisotropic turbulence. The LSS spectrum represents the dominant portion of acoustic energy in the downstream direction of the jet flow-field. Recall that combinations of the LSS and SSS spectra represent the total noise from the fine-scale and large-scale noise sources.

Compare the relative spectral widths of the present prediction with that of the LSS of Tam and Seiner Tam et al. (1996) in Fig. 15. The predicted spectrum is much narrower than the LSS spectrum. This is due to the use of a flow-field that is a two-dimensional shear layer while the LSS is calibrated against jet turbulence. The two-dimensional shear layer has much higher spatial coherence because very strong paired vortices (as can be seen in our LES figures) exist that do not break down as readily as jet turbulence. The major advantage of this theory, when applied to the two-dimensional shear layer, is that we are able to explicitly study the large-scale coherent anisotropic turbulence. The two-dimensional shear layer has the majority of its energy contained in the paired vortices.
Summary and Conclusion

We decomposed the Navier-Stokes equations into a base flow, fluctuations involving large-scale anisotropic turbulence, and the associated radiated noise. A closed-form solution for the radiating acoustic waves is devised and the source terms are modeled based on a new instability wave model. The instability waves and the interaction of instability waves with other instability waves are responsible for the dominant acoustic radiation. We applied our method to the two-dimensional shear layer and compared our predictions with those of Tam and Morris, and the agreement is very good. We captured both the dominant radiation in the downstream direction, and the secondary broadband radiation at slightly higher angles relative to the dominant radiation angle. The model depends on estimation of the instability waves strength and also the wavenumber spectra of each of the field-variables. These are found through both LES and steady RANS CFD solutions. One of the major advantages of this particular approach and the examination of the two-dimensional shear layer is that the majority of the energy of the flow and associated acoustic radiation is contained within large-scale spatially coherent anisotropic structures. Research is already being conducted to extend this model to a more general framework that includes both anisotropic sources and locally isotropic sources that includes their interaction with shock waves.

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Personal Statement (SAEM)

Boeing / A. D. Welliver Professor of Aerospace Engineering Philip J. Morris wrote his Ph.D. dissertation on, ‘The Structure of Turbulent Shear Flow’ (see Morris (1971)). His advisor was the esteemed Professor Geoffrey M. Lilley. Throughout Professor Morris’s career, he focused on turbulent flows and their acoustic radiation with a particular emphasis on jet aeroacoustics. His dissertation under Professor Lilley set the stage for the rest of his career. Professor Morris wrote numerous articles on turbulent shear layers and their acoustic radiation, and these are reviewed within this paper. Dr. Morris’s research on shear layer turbulence and aeroacoustics is the inspiration for the present paper.

I had the pleasure of traveling to University of Southampton as a Ph.D. candidate with my advisor, Professor Morris, to visit the Institute of Sound and Vibration. After the two-day meeting, Professor Morris and myself traveled to meet with Professor Lilley. It was an opportunity of a lifetime to share a pint of beer at an English pub in the New Forrest with two world renowned aeroacousticians. This meeting was one of many examples representing the special bond between a Professor and his Ph.D. student.

The academic advisee-advisor relationship represents more than producing research, a dissertation, and a Ph.D. For example, Professor Morris taught this student how to think independently, how to formulate problems, how to learn how to learn, the value of research, and a strong work ethic. I hope that I can incorporate these lessons and teach them to my students within my own professorship. Thank you, Professor Morris, for the values you have given your students.
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Appendix: Source Terms

The continuity-continuity source term is

\[
\{ R_{(0),(0)} \} = \left\{ \begin{array}{c}
\left( \frac{\hat{\rho}(1) + \bar{\rho}(1)}{l_{x,(1)}} \right) (\hat{u}(1) + \bar{u}(1)) - \left( \frac{\hat{\rho}(1) + \bar{\rho}(1)}{l_{y,(1)}} \right) (\hat{v}(1) + \bar{v}(1)) - \frac{\hat{\rho}(1) + \bar{\rho}(1)}{\tau(1)} \\
\times \left[ \left( \frac{\hat{\rho}(2) + \bar{\rho}(2)}{l_{x,(2)}} \right) (\hat{u}(2) + \bar{u}(2)) - \left( \frac{\hat{\rho}(2) + \bar{\rho}(2)}{l_{y,(2)}} \right) (\hat{v}(2) + \bar{v}(2)) - \frac{\hat{\rho}(2) + \bar{\rho}(2)}{\tau(2)} \right]. \end{array} \right.
\]

(26)

The continuity-momentum source term in the \( x \) direction is

\[
\{ R_{(0),(1)} \} = \left\{ \begin{array}{c}
\left( \frac{\hat{\rho}(1) + \bar{\rho}(1)}{l_{x,(1)}} \right) (\hat{u}(1) + \bar{u}(1)) - \left( \frac{\hat{\rho}(1) + \bar{\rho}(1)}{l_{y,(1)}} \right) (\hat{v}(1) + \bar{v}(1)) - \frac{\hat{\rho}(1) + \bar{\rho}(1)}{\tau(1)} \\
\times \left[ \frac{4\mu(\hat{u}(2) + \bar{u}(2))^2}{3l_{x,(2)}^2} + \frac{\mu(\hat{v}(2) + \bar{v}(2))^2}{3l_{y,(2)}l_{x,(2)}} - \frac{(\hat{\rho}(2) + \bar{\rho}(2))(\hat{u}(2) + \bar{u}(2))^2}{l_{x,(2)}} - \frac{(\hat{\rho}(2) + \bar{\rho}(2))(\hat{v}(2) + \bar{v}(2))^2}{l_{y,(2)}} \right]. \end{array} \right.
\]

(27)

The continuity-momentum source term in the \( y \) direction is

\[
\{ R_{(0),(2)} \} = \left\{ \begin{array}{c}
\left( \frac{\hat{\rho}(1) + \bar{\rho}(1)}{l_{x,(1)}} \right) (\hat{u}(1) + \bar{u}(1)) - \left( \frac{\hat{\rho}(1) + \bar{\rho}(1)}{l_{y,(1)}} \right) (\hat{v}(1) + \bar{v}(1)) - \frac{\hat{\rho}(1) + \bar{\rho}(1)}{\tau(1)} \\
\times \left[ \frac{\mu(\hat{v}(2) + \bar{v}(2))^2}{3l_{x,(2)}^2} + \frac{\mu(\hat{u}(2) + \bar{u}(2))^2}{3l_{y,(2)}l_{y,(2)}} + \frac{4\mu(\hat{v}(2) + \bar{v}(2))^2}{3l_{y,(2)}^2} - \frac{(\hat{\rho}(2) + \bar{\rho}(2))(\hat{v}(2) + \bar{v}(2))^2}{l_{y,(2)}} \right. \\
- \left. \frac{(\hat{\rho}(2) + \bar{\rho}(2))(\hat{u}(2) + \bar{u}(2))(\hat{v}(2) + \bar{v}(2))}{l_{x,(2)}} - \frac{(\hat{\rho}(2) + \bar{\rho}(2))(\hat{v}(2) + \bar{v}(2))^2}{l_{y,(2)}} - \frac{(\hat{\rho}(2) + \bar{\rho}(2))(\hat{u}(2) + \bar{u}(2))(\hat{v}(2) + \bar{v}(2))}{l_{x,(2)}} \right]. \end{array} \right.
\]

(28)

The continuity-energy source term is

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The streamwise momentum-momentum source term is.

\[ \{ R_{(0),(3)} \} = \left( \frac{-\hat{R}_x(1) + \hat{p}_x(1)}{l_x(1)} - \frac{(\hat{p}_x(1) + \hat{p}_y(1))}{l_y(1)} - \frac{\hat{p}_y(1) + \hat{p}_y(1)}{\tau(1)} \right) \]

\[ \times \left( (\gamma - 1)c_p\mu \left( \frac{\tilde{T}(2) + \tilde{T}(2)}{l_x(2)} \right) + \frac{\tilde{T}(2) + \tilde{T}(2)}{l_y(2)} \right) \]

\[ + (\gamma - 1) \mu \left( \frac{\tilde{T}(2) + \tilde{T}(2)}{l_x(2)} \right) + \frac{\tilde{T}(2) + \tilde{T}(2)}{l_y(2)} \right) \]

\[ - \frac{2}{3} \mu \left( \frac{(\tilde{u}(2) + \tilde{u}(2))}{l_x(2)} \right) + \frac{(\tilde{u}(2) + \tilde{u}(2))(\tilde{v}(2) + \tilde{v}(2))}{l_y(2)} \right) \]

\[ + (\gamma - 1) \mu \left( \frac{(\tilde{v}(2) + \tilde{v}(2))^2}{l_x(2)} \right) + \frac{(\tilde{u}(2) + \tilde{u}(2))(\tilde{v}(2) + \tilde{v}(2))}{l_y(2)} \right) \]

\[ - \frac{2}{3} \mu \left( \frac{(\tilde{u}(2) + \tilde{u}(2))}{l_x(2)} \right) + \frac{(\tilde{u}(2) + \tilde{u}(2))(\tilde{v}(2) + \tilde{v}(2))}{l_y(2)} \right) \]

\[ - \gamma \left( \frac{(\hat{p}_x(2) + \hat{p}_y(2))(\tilde{u}(2) + \tilde{u}(2))}{l_y(2)} \right) + \left( \frac{\hat{p}_x(2) + \hat{p}_y(2)(\tilde{v}(2) + \tilde{v}(2))}{l_x(2)} \right) \]

\[ - \frac{1}{2}(\gamma - 1) \left( \frac{(\hat{p}_x(2) + \hat{p}_y(2))(\tilde{u}(2) + \tilde{u}(2))^3}{l_x(2)} \right) + \left( \frac{\hat{p}_x(2) + \hat{p}_y(2)(\tilde{v}(2) + \tilde{v}(2))}{l_y(2)} \right) \]

\[ - \frac{\hat{p}_y(2) + \hat{p}_y(2)}{\tau(2)} - \frac{1}{2}(\gamma - 1) \left( \frac{(\hat{p}_x(2) + \hat{p}_y(2))(\tilde{u}(2) + \tilde{u}(2))^2}{\tau(2)} \right) + \left( \frac{\hat{p}_x(2) + \hat{p}_y(2)(\tilde{v}(2) + \tilde{v}(2))^2}{\tau(2)} \right) \right). \]
\[
\{ R_{(1),(1)} \} = \left( \frac{4\mu(\bar{u}(1) + \bar{u}(1))^2}{3l_x(1)^2} + \frac{\mu(\bar{v}(1) + \bar{v}(1))^2}{3l_x(1)l_y(1)} - \frac{\bar{p}(1) + \bar{p}(1)}{l_x(1)} - \frac{(\bar{\rho}(1) + \bar{\rho}(1))(\bar{u}(1) + \bar{u}(1))^2}{l_x(1)} \right.
\]
\[
+ \frac{\mu(\bar{u}(1) + \bar{u}(1))^2}{l_y(1)} - \frac{(\bar{\rho}(1) + \bar{\rho}(1))(\bar{u}(1) + \bar{u}(1))}{\tau(1)} - \frac{(\bar{\rho}(1) + \bar{\rho}(1))(\bar{u}(1) + \bar{u}(1))}{\tau(1)} \right)
\]
\[
\times \left( \frac{4\mu(\bar{u}(2) + \bar{u}(2))^2}{3l_x(2)^2} + \frac{\mu(\bar{v}(2) + \bar{v}(2))^2}{3l_x(2)l_y(2)} - \frac{\bar{p}(2) + \bar{p}(2)}{l_x(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{u}(2) + \bar{u}(2))^2}{l_x(2)} \right.
\]
\[
+ \frac{\mu(\bar{u}(2) + \bar{u}(2))^2}{l_y(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{u}(2) + \bar{u}(2))}{\tau(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{u}(2) + \bar{u}(2))}{\tau(2)} \right)
\]
\[
\times \left( \frac{4\mu(\bar{v}(2) + \bar{v}(2))^2}{3l_y(2)^2} + \frac{\mu(\bar{v}(2) + \bar{v}(2))^2}{3l_x(2)l_y(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{v}(2) + \bar{v}(2))^2}{l_x(2)} \right)
\]
\[
= \left( \frac{4\mu(\bar{u}(1) + \bar{u}(1))^2}{3l_x(1)^2} + \frac{\mu(\bar{v}(1) + \bar{v}(1))^2}{3l_x(1)l_y(1)} - \frac{\bar{p}(1) + \bar{p}(1)}{l_x(1)} - \frac{(\bar{\rho}(1) + \bar{\rho}(1))(\bar{u}(1) + \bar{u}(1))^2}{l_x(1)} \right.
\]
\[
\times \left( \frac{4\mu(\bar{u}(2) + \bar{u}(2))^2}{3l_x(2)^2} + \frac{\mu(\bar{v}(2) + \bar{v}(2))^2}{3l_x(2)l_y(2)} - \frac{\bar{p}(2) + \bar{p}(2)}{l_x(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{u}(2) + \bar{u}(2))^2}{l_x(2)} \right)
\]
\[
\times \left( \frac{4\mu(\bar{v}(2) + \bar{v}(2))^2}{3l_y(2)^2} + \frac{\mu(\bar{v}(2) + \bar{v}(2))^2}{3l_x(2)l_y(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{v}(2) + \bar{v}(2))^2}{l_y(2)} \right) \cdot \tau(2). \tag{30}\]

The streamwise momentum-momentum cross-stream source term is

\[
\{ R_{(1),(2)} \} = \left( \frac{4\mu(\bar{u}(1) + \bar{u}(1))^2}{3l_x(1)^2} + \frac{\mu(\bar{v}(1) + \bar{v}(1))^2}{3l_x(1)l_y(1)} - \frac{\bar{p}(1) + \bar{p}(1)}{l_x(1)} - \frac{(\bar{\rho}(1) + \bar{\rho}(1))(\bar{u}(1) + \bar{u}(1))^2}{l_x(1)} \right.
\]
\[
+ \frac{\mu(\bar{u}(1) + \bar{u}(1))^2}{l_y(1)} - \frac{(\bar{\rho}(1) + \bar{\rho}(1))(\bar{u}(1) + \bar{u}(1))}{\tau(1)} - \frac{(\bar{\rho}(1) + \bar{\rho}(1))(\bar{u}(1) + \bar{u}(1))}{\tau(1)} \right)
\]
\[
\times \left( \frac{4\mu(\bar{u}(2) + \bar{u}(2))^2}{3l_x(2)^2} + \frac{\mu(\bar{v}(2) + \bar{v}(2))^2}{3l_x(2)l_y(2)} - \frac{\bar{p}(2) + \bar{p}(2)}{l_x(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{u}(2) + \bar{u}(2))^2}{l_x(2)} \right)
\]
\[
+ \frac{\mu(\bar{u}(2) + \bar{u}(2))^2}{l_y(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{u}(2) + \bar{u}(2))}{\tau(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{u}(2) + \bar{u}(2))}{\tau(2)} \right)
\]
\[
\times \left( \frac{4\mu(\bar{v}(2) + \bar{v}(2))^2}{3l_y(2)^2} + \frac{\mu(\bar{v}(2) + \bar{v}(2))^2}{3l_x(2)l_y(2)} - \frac{(\bar{\rho}(2) + \bar{\rho}(2))(\bar{v}(2) + \bar{v}(2))^2}{l_y(2)} \right) \cdot \tau(2). \tag{31}\]

The streamwise momentum-energy equation is

\[
\text{Prepared using sagej.cls} \]
\begin{align}
\{ R_{(1),(3)} \} &= \left( \frac{4\mu(\bar{u}(1) + \bar{u}(1))^2}{3l_{x(1)}^2} + \frac{\mu(\bar{v}(1) + \bar{v}(1))^2}{3l_{x(1)}l_{y(1)}} - \frac{\bar{p}(1) + \bar{p}(1)}{l_{x(1)}} - \frac{(\bar{p}(1) + \bar{p}(1))(\bar{u}(1) + \bar{u}(1))^2}{\tau_{(1)}} \right) \\
&\quad + \mu(\bar{u}(1) + \bar{u}(1))^2 \left( \frac{\bar{p}(1) + \bar{p}(1)}{l_{y(1)}} \right) - \frac{(\bar{p}(1) + \bar{p}(1))(\bar{u}(1) + \bar{u}(1))(\bar{v}(1) + \bar{v}(1))}{l_{x(1)}} - \frac{(\bar{p}(1) + \bar{p}(1))(\bar{u}(1) + \bar{u}(1))}{\tau_{(1)}} \\
&\quad \times \left( \frac{(\gamma - 1)\mu(\bar{u}(2) + \bar{v}(2))(\bar{v}(2) + \bar{v}(2))}{p_r} \right) + (\gamma - 1)\mu \left( \frac{(\bar{u}(2) + \bar{u}(2))(\bar{v}(2) + \bar{v}(2))}{l_{x(2)}} + \frac{(\bar{u}(2) + \bar{u}(2))(\bar{v}(2) + \bar{v}(2))}{l_{y(2)}} \right) \\
&\quad + (\gamma - 1)\mu \left( \frac{(\bar{u}(2) + \bar{u}(2))(\bar{v}(2) + \bar{v}(2))}{l_{x(2)}} + \frac{(\bar{u}(2) + \bar{u}(2))(\bar{v}(2) + \bar{v}(2))}{l_{y(2)}} \right) \\
&\quad - \frac{2}{3}\mu \left( \frac{(\bar{u}(2) + \bar{v}(2))(\bar{v}(2) + \bar{v}(2))}{l_{x(2)}} + \frac{(\bar{u}(2) + \bar{v}(2))(\bar{v}(2) + \bar{v}(2))}{l_{y(2)}} \right) \\
&\quad - \frac{1}{2}(\gamma - 1) \left( \frac{(\bar{p}(2) + \bar{p}(2))(\bar{u}(2) + \bar{u}(2))^3}{l_{x(2)}} + \frac{(\bar{p}(2) + \bar{p}(2))(\bar{u}(2) + \bar{u}(2))(\bar{v}(2) + \bar{v}(2))^2}{l_{x(2)}} \right) \\
&\quad + \frac{(\bar{p}(2) + \bar{p}(2))(\bar{u}(2) + \bar{u}(2))^2(\bar{v}(2) + \bar{v}(2))}{l_{y(2)}} + \frac{(\bar{p}(2) + \bar{p}(2))(\bar{v}(2) + \bar{v}(2))^3}{l_{y(2)}} - \frac{\bar{p}(2) + \bar{p}(2)}{\tau_{(2)}} \right) \\
&\quad - \frac{1}{2}(\gamma - 1) \left( \frac{(\bar{p}(2) + \bar{p}(2))(\bar{u}(2) + \bar{u}(2))^2}{\tau_{(2)}} + \frac{(\bar{p}(2) + \bar{p}(2))(\bar{v}(2) + \bar{v}(2))^2}{\tau_{(2)}} \right). \quad (32)
\end{align}

The cross-stream momentum-momentum source term is
\[
\{ R_{(2),(2)} \} = \left( \frac{\mu(\ddot{v}_1 + \overline{v}_1)^2}{l_{x,(1)}^2} + \frac{\mu(\ddot{u}_1 + \overline{u}_1)^2}{3l_{x,(1)}l_{y,(1)}} - \frac{(\ddot{\rho}_1 + \overline{\rho}_1)(\ddot{u}_1 + \overline{u}_1)(\ddot{v}_1 + \overline{v}_1)}{l_{x,(1)}} \right) \\
+ \frac{4\mu(\ddot{v}_1 + \overline{v}_1)^2}{3l_{y,(1)}^2} - \frac{\ddot{\rho}_1 + \overline{\rho}_1}{l_{y,(1)}} - \left( \frac{(\ddot{\rho}_1 + \overline{\rho}_1)(\ddot{v}_1 + \overline{v}_1)^2}{l_{y,(1)}} - \frac{(\ddot{\rho}_1 + \overline{\rho}_1)(\ddot{v}_1 + \overline{v}_1)}{\tau_{(1)}} \right) \\
\times \left( \frac{\mu(\ddot{v}_2 + \overline{v}_2)^2}{l_{x,(2)}^2} + \frac{\mu(\ddot{u}_2 + \overline{u}_2)^2}{3l_{x,(2)}l_{y,(2)}} - \frac{(\ddot{\rho}_2 + \overline{\rho}_2)(\ddot{u}_2 + \overline{u}_2)(\ddot{v}_2 + \overline{v}_2)}{l_{x,(2)}} \right) \\
+ \frac{4\mu(\ddot{v}_2 + \overline{v}_2)^2}{3l_{y,(2)}^2} - \frac{\ddot{\rho}_2 + \overline{\rho}_2}{l_{y,(2)}} - \left( \frac{(\ddot{\rho}_2 + \overline{\rho}_2)(\ddot{v}_2 + \overline{v}_2)^2}{l_{y,(2)}} - \frac{(\ddot{\rho}_2 + \overline{\rho}_2)(\ddot{v}_2 + \overline{v}_2)}{\tau_{(2)}} \right) \\
\times \left( \frac{(\gamma - 1)c_p\mu}{\mathcal{P}_r} \left( \frac{T_{(2)} + \mathcal{T}_{(2)}}{l_{x,(2)}^2} + \frac{T_{(2)} + \mathcal{T}_{(2)}}{l_{y,(2)}^2} \right) + (\gamma - 1)\mu \left( \frac{(\ddot{u}_2 + \overline{u}_2)(\ddot{v}_2 + \overline{v}_2)}{l_{x,(2)}} \right) + \frac{2(\ddot{u}_2 + \overline{u}_2)^2}{l_{x,(2)}^2} \right) \\
- \left( \frac{\ddot{u}_2 + \overline{u}_2)^2}{l_{x,(2)}^2} + \frac{2(\ddot{v}_2 + \overline{v}_2)^2}{l_{y,(2)}^2} \right) - \frac{2}{3}\mu \left( \frac{(\ddot{u}_2 + \overline{u}_2)^2}{l_{x,(2)}^2} + \frac{(\ddot{v}_2 + \overline{v}_2)^2}{l_{y,(2)}^2} \right) - \frac{2}{3}\mu \left( \frac{(\ddot{u}_2 + \overline{u}_2)(\ddot{v}_2 + \overline{v}_2)}{l_{x,(2)}l_{y,(2)}} \right) + \frac{2(\ddot{v}_2 + \overline{v}_2)^2}{l_{y,(2)}^2} \\
- \gamma \left( \frac{\ddot{u}_2 + \overline{u}_2)^2}{l_{x,(2)}^2} + \frac{2(\ddot{v}_2 + \overline{v}_2)^2}{l_{y,(2)}^2} \right) - \frac{1}{2}(\gamma - 1) \left( \frac{(\ddot{\rho}_2 + \overline{\rho}_2)(\ddot{u}_2 + \overline{u}_2)^3}{l_{x,(2)}} \right) \\
+ \frac{(\ddot{\rho}_2 + \overline{\rho}_2)(\ddot{u}_2 + \overline{u}_2)(\ddot{v}_2 + \overline{v}_2)^2}{l_{x,(2)}l_{y,(2)}} + \frac{(\ddot{\rho}_2 + \overline{\rho}_2)(\ddot{u}_2 + \overline{u}_2)^2(\ddot{v}_2 + \overline{v}_2)}{l_{x,(2)}} \\
+ \frac{(\ddot{\rho}_2 + \overline{\rho}_2)(\ddot{v}_2 + \overline{v}_2)^3}{l_{y,(2)}} - \frac{\ddot{\rho}_2 + \overline{\rho}_2}{\tau_{(2)}} - \frac{1}{2}(\gamma - 1) \left( \frac{(\ddot{\rho}_2 + \overline{\rho}_2)(\ddot{u}_2 + \overline{u}_2)^2}{\tau_{(2)}} \right) \\
+ \frac{(\ddot{\rho}_2 + \overline{\rho}_2)(\ddot{v}_2 + \overline{v}_2)^2}{\tau_{(2)}} \right). \\
\]
\[
\{ R_{(3), (3)} \} = \left\{ \begin{array}{c}
\frac{(\gamma - 1) e_{\mu} \left( \frac{T_1^{(1)} + T_2^{(1)}}{y_{x, (1)}} + \frac{T_2^{(1)} + T_3^{(1)}}{y_{y, (1)}} \right)}{\rho_r} + (\gamma - 1) \mu \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} + \frac{2(\hat{u}_{(1)} + \bar{v}_{(1)})^2}{t_{y, (1)}^2} \right)
+ \left( \frac{1}{2} \right) \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} \right) - \frac{2}{3} \mu \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} + \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{y, (1)}^2} \right) \\
+ \left( \frac{1}{2} \right) \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} \right) - \frac{2}{3} \mu \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} + \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{y, (1)}^2} \right) \\
+ \left( \frac{1}{2} \right) \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} \right) - \frac{2}{3} \mu \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} + \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{y, (1)}^2} \right) \\
+ \left( \frac{1}{2} \right) \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} \right) - \frac{2}{3} \mu \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} + \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{y, (1)}^2} \right) \\
+ \left( \frac{1}{2} \right) \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} \right) - \frac{2}{3} \mu \left( \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{x, (1)}^2} + \frac{(\hat{u}_{(1)} + \bar{v}_{(1)}) \hat{v}_{(1)} + \bar{v}_{(1)}}{t_{y, (1)}^2} \right) \\
\end{array} \right. \\
\right) \right\}.
\]
Tables

Table 1. LES parameters for each solution. Note $\Delta t$ is non-dimensional. Run four is for sampling data.

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<th>Order</th>
</tr>
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<td>8e6</td>
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</tr>
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<td>2e-6</td>
<td>2e6</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>2e-6</td>
<td>2e6</td>
<td>3</td>
</tr>
</tbody>
</table>
Figures

Figure 1. The computational grid used for the steady RANS solutions. The boundary conditions are labeled.
Figure 2. The computational grid used for the LES.
Figure 3. The boundary conditions used in the LES.
Figure 4. Steady RANS values of $k$ and $\epsilon$ along the $x$-axis.
Figure 5. Estimated integral length and time scales along the $x$-axis.
Figure 6. Vorticity magnitude with gray scale of pressure plot.
Figure 7. Contours of $\rho$ for $M = 0.75$ from LES.
Figure 8. Streamwise velocity distributions generated from LES at $M = 0.75$ compared with Tam.
Figure 9. The wavenumber energy spectrum, $E_u$, at streamwise location $x = 0.0446$ m and $y = 0$ m for $\mathcal{M} = 0.75$. The Kolmogorov -5/3 power law is superimposed as a dashed line.
Figure 10. Instability wave through central axis in direction of flow at various decay times including the maximum envelope.
Figure 11. Example isolated instability wave.
Figure 12. Example interaction of two-instability waves.
Figure 13. Comparison of directivity predictions based on LES and steady RANS arguments with those of Tam and Morris at three Mach numbers.
Figure 14. Predictions at $M = 1.75$ at various non-dimensional frequencies referenced to the peak frequency.
Figure 15. Comparison of the LES based prediction with the large-scale and fine-scale similarity spectra of Tam et al. (1996).